Application of MHPM for Solving Semi-Linear Fractional Differential Equations

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Abstract. In this paper, we have introduced a general class of semi-linear first-order fractional differential equations (FrDEs) which is modified mathematically in many problems in many significant fields of applied science and engineering. The homotopy perturbation method (HPM) has been modified for solving generalized linear first-order FrDEs. Also, we have tested the modified homotopy perturbation method (MHPM) on the solving of different implementations of problems which show the accuracy and efficiency of the proposed method. The approximated solutions of the tested problems using MHPM are agree well with analytical solutions. However, these solutions proved that the proposed method to be high accurate and efficient method. MHPM has been used for solving class of generalized factional Riccati DE. The approximated solution of a class of nonlinear Riccati FrDE has been studied. Also, we have tested the MHPM on the solving of different implementations of this class which are show the accuracy and efficiency of the proposed method. The approximated solutions using proposed MHPM of the tested problems are agree well with analytical solutions and they proved to be more accurate.

Key words: Homotopy, HPM, MHPM, Fractional Differential Equations, FrDEs.

Introduction

The differential equations (DEs) are the most important tools in mathematical model for the phenomena of physics. Most of mathematical models in branches of applied science and engineering are expressed in terms of functions in one or more than one of variables and their derivatives. Fractional calculus is a mathematical tool and is a generalization for the integer order calculus. However, the applications of fractional calculus have recently been investigated. Many mathematical models in physics and engineering systems can be elegantly modeled with the tools of the fractional derivative, such as viscoelastic systems, dielectric electrolyte. electrolyte polarization, and polarization, (Wang et al., 2014: 66). The researchers: Mechee and Kadhim (2016a: 4687-4715, 2016b: 1452-1460), Mechee et al. (2014: 6, 2016: 3959-3975) developed numerical methods for solving ODEs of third-, fourth- and fifth-order. Due to the increasing in the applications of FrDEs, there has been important interest in developing numerical integrators for the solution of FrDEs. There are some modern methods for solving DEs like as domain decomposition method, generalized differential transform method, variational iteration method, finite difference method and wavelet method. Nowadays, some of algorithms for the numerical solutions of FrDEs have been proposed in recent years. There are some published papers in literature review of the methods of solutions of mathematical models which contains DEs. Some researchers developed the numerical and analytical methods for solving DEs. Podlubny (1998) used Laplace transform method to solve the FrPDEs with constant coefficients, Odibat and Momani (2008: 194) applied generalized differential transform method to solve the linear FrPDEs numerically, Zhang (2009) discussed a practical implicit method to solve a class of initial boundary value (IVP) of time fractional convection diffusion equations with variable coefficients, Ahmad and Sprott (2003: 339) studied the chaos in autonomous nonlinear systems of FrDEs, Mechee and Senu (2012: 851) introduced the solution of Lane-Emden FrDE by least square method and collocation method numerically, Saeed and Rehman (2015: 630) used Haar wavelet (HW) Picard method for nonlinear FrPDEs, Shiralashetti and Deshi (2016: 47) introduced HW collocation method for solving ordinary and fractional Riccati equation, Mechee et al. (2019) have used HW technique to approximate the solutions of DEs of fractional-order, Saeed and Rehman (2013: 630) used Haar wavelet quasi linearization technique for fractional nonlinear DEs while Wang et al. (2014: 66) introduced HW method for solving fractional PDEs numerically. For HPM, Mechee et al. (2017b) have studied the general solution of second-order PDEs using HPM, Mechee et al. (2017a: 2527) have modified HPM for solving generalized linear CDEs and Mechee and Al-Juaifri (2019) used HPM for solving generalized Riccati DE. In this paper, we have developed HPM and studied the approximated solutions of FrDEs using the MHPM.

Preliminary

Analysis of Homotopy Perturbation Method (HPM)

In this subsection, we present the algorithm of HPM to illustrate the basic steps of the HPM. Consider the following differential equation (Abbasbandy (2006: 581), Batiha (2015: 24), Chun and Sakthivel (2010) and Neamaty and Darzi (2010: 317-369):

$$T(w(t)) - g\,(\mathrm{t}) = 0, \mathrm{t} \in \Omega \tag{1}$$

with boundary condition

$$C\left(w(t), \frac{\partial w(t)}{\partial t}\right) = 0, \qquad t \in \partial\,\Omega \tag{2}$$

where T = differential operator, $g(t)$ = given analytic function, C = boundary operator and $\partial\,\Omega$= the boundary of the domain Ω. T operator can be generally written as two parts of L and N where the linear part is L while the nonlinear part in the DE is N, Therefore Equation (1) can be rewritten as follows (He, 1999: 257):

$$L\big(w(t)\big) + N\big(w(t)\big) - g(t) = 0, \tag{3}$$

One can construct the following homotopy using homotopy technique, $V\big((t);\, p\big)$: $\Omega \times [0;1] \longrightarrow R$ which satisfies:

$$H(v(t);p) = (1-p)[Lv(t) - Lw_0(t)] + p[Lv(t) + Nv(t) - g(t)] = 0; \tag{4}$$

or

$$H(v(t);p) = Lv(t) - Lw_0(t) + pLw_0(t) + p[Nv(t) - g(t)] = 0, \tag{5}$$

where $p \in [0;1]$; $t \in \Omega$ and the homotopy parameter is p while the initial approximation for the solution of equation (1) which satisfies the boundary conditions is w_0. Using Equation (4) or (5), we obtain the following equation:

$$H(v(t);0) = Lv(t) - Lw_0(t) = 0; \tag{6}$$

and

$$H(v(t);1) = Lv(t) + Nv(t) - g(t) = 0; \tag{7}$$

Suppose the solution of equations (4) or (5) can be written as a series in *p* as follows:

$$v(t) = v_0(t) + pv_1(t) + p^2v_2(t) + p^3v_3(t) + \cdots = \sum_{i=0}^{\infty} p^i v_i(t), \tag{8}$$

set $p \longrightarrow 1$ results in the approximate solution of (1). Consequently,

$$w(t) = \lim_{p \to 1} v(t) = v_0(t) + v_1(t) + v_2(t) + v_3(t) + \cdots = \sum_{i=0}^{\infty} v_i(t), \tag{9}$$

It's important to note that the major advantage of HPM is the perturbation equation can be constructed freely in different ways and the approximation can be freely selected.

Semi-Linear Fractional Differential Equations

In this subsection, we have considered a class of FrDEs which have the following form:

$$D_*^{\alpha} w(t) + g\big(t; w(t)\big) = f(t);\ t > 0;\ m - 1 < \alpha \leq m; \tag{10}$$

subject to the initial conditions

$$w^{(n)}(0) = a_n; \quad n = 0,1, \dots, m-1, \tag{11}$$

where a_n for $n = 0,1, \dots, m-1,$ are arbitrary constants.

Analysis of MHPM for Solving Fractional Differential Equations

The objectives of this paper are to develop HPM and to study the approximated solutions of the semi-linear FrDEs in Equation (10) according the following algorithm.

Algorithm of MHPM

In this subsection, we have presented the algorithm of the new modification of the HPM. To illustrate the basic ideas of the MHPM as the following algorithm:

1. Construct a homotopy for FrDEs in Equation (10) as follow:
2.

$$w^{(m)}(t) - f(t) = p[w^{(m)}(t) - D_*^{\alpha}(w(t)) - g(t; w(t))]; \quad p \longrightarrow [0;1]: \tag{12}$$

3. By substituting Equation (12) into Equation (10) and equating the identical powers *p*,
4. Consequently, we substitute $w(t) = \sum_{i=0}^{\infty} p^i w_i(t),$ in the Equation (12) and, using expansion for the functions $g(t; w(t))$, for the coefficients functions $g\big(t; w(t)\big) = \sum_{j=0}^{n} A_j w^j(t)$.
5. Hence, consider the solution of Equation (10) as the following form:

$$w(t) = w_0(t) + pw_1(t) + p^2w_2(t) + p^3w_3(t) + \cdots \tag{13}$$

5. Collect the identical powers of *p* then, we have the following equations:

$$p^0 : w_0^{(m)}(t) = g(t), \tag{14}$$

$$p^1 : w_1^{(m)}(t) = w_0^{(m)}(t) - D_*^{\alpha} w(t)_0 - \sum_{j=0}^{n} A_j w_0^j(t), \quad (15)$$

$$p^2 : w_2^{(m)}(t) = w_1^{(m)}(t) - D_*^{\alpha} w(t)_1 - w_1(t) \sum_{j=1}^{n} j A_j w_0^{j-1}(t), \quad (16)$$

$$p^3 : w_3^{(m)}(t) = w_2^{(m)}(t) - D_*^{\alpha} w(t)_2 - w_2(t) \sum_{j=1}^{n} j A_j w_0^{j-1}(t) + w_1^2(t) \sum_{j=2}^{n} j\, \alpha_j A_j w_0^{j-2}(t), \quad (17)$$

$$p^4 : w_4^{(m)}(t) = w_3^{(m)}(t) - D_*^{\alpha} w(t)_3 - w_3(t) \sum_{j=1}^{n} j A_j w_0^{j-1}(t) + w_1(t) w_2(t) \sum_{j=2}^{n} (j^2 - j) A_j w_0^{j-2}(t) + w_1^3(t) \sum_{j=3}^{n} j\, \beta_i A_i w_0^{j-3}(t), \quad (18)$$

and

$$p^5 : w_5^{(m)}(t) = w_4^{(m)}(t) - D_*^{\alpha} w(t)_4 - w_4(t) \sum_{j=1}^{n} j A_j w_0^{j-1}(t) + w_1(t) w_3(t) \sum_{j=2}^{n} (j^2 - j) A_j w_0^{j-2}(t) + w_2^2(t) \sum_{j=2}^{n} j\, \beta_i A_j w_0^{j-2}(t) + w_1^2(t) w_2(t) \sum_{j=3}^{n} \sigma_j A_j w_0^{j-3}(t) + w_1^4(t) \sum_{j=4}^{n} \omega_j A_j w_0^{j-4}(t). \quad (19)$$

Where $\alpha_i = 1,3,6,10,15, \ldots$; $\beta_i = 1,4,10,20, \ldots \ldots$, $\sigma_i = 3,12,30,60, \ldots$ *and* $\omega_i = 1,5,15, \ldots \ldots$,

6. However, from the equations (14) -(19), we can get the general solution of Equation (10) as follows:

$$w(t) = w_0(t) + w_1(t) + w_2(t) + w_3(t) + \cdots \quad (20)$$

Implementation

For evaluating the solution of fractional differential equations using MHPM, we have provided some different examples for comparison study with analytical solutions of these examples.

Example 1. The Crisp FrDE

Consider the Crisp FrDE as follows:

$$D_*^{\alpha} w(t) + w(t) = 0; \quad 0 < a \leq 1; \quad (21)$$

with the initial conditions w (0) = a:

Using the algorithm of MHPM for Equation (21), we have the following terms:

$$w_0(t) = a; \quad (22)$$

$$w_k(t) = \int \left(w_{k-1}(t) - D_*^{\alpha} w_{k-1}(t) - w_{k-1}(t) \right) dt \,; \quad k = 1,2,3, \ldots \quad (23)$$

So, simplification of equation (23) lends to the following solutions:

$$w_1(t) = -at, \quad (24)$$

$$w_2(t) = -at + \frac{at^{2-\alpha}}{(2-\alpha)!} + \frac{at^2}{2!}, \quad (25)$$

$$w_{3}(t) = -at + \frac{2at^{2-\alpha}}{(2-\alpha)!} + at^2 - \frac{2at^{3-\alpha}}{(3-\alpha)!} - \frac{at^3}{3!} - \frac{at^{3-2\alpha}}{(3-2\alpha)!}, \qquad (26)$$

Then, the approximated solution is given as follows:

$$w(t) = w_0(t) + w_1(t) + w_2(t) + w_3(t) + \cdots \qquad (27)$$

$$w(t) = a - 3at + \frac{at^{2-\alpha}}{(2-\alpha)!} + \frac{at^2}{2!} + \frac{2at^{2-\alpha}}{(2-\alpha)!} + at^2 - \frac{2at^{3-\alpha}}{(3-\alpha)!} - \frac{at^3}{3!} - \frac{at^{3-2\alpha}}{(3-2\alpha)!} + \ldots \qquad (28)$$

if $\alpha = 1$ then, the general solution is given as follow:

$$w(t) = ae^{-t}. \qquad (29)$$

Example 2. Riccati FrDE
Consider the Riccati FrDE as follows*:*

$$D_*^{\alpha} w(t) + w^2(t) = 0; \quad 0 < a \leq 1; \qquad (30)$$

with the initial condition $w(0) = 0$, comparing Equation (30), we have the following form

$$w_{0}(t) = t; \qquad (31)$$

$$w_{k}(t) = \int \left(w_{k-1}(t) - D_*^{\alpha} w_{k-1}(t) - \sum_{i=0}^{k-1} w_{i}(t) w_{k-1-i}(t)\right) dt; \qquad (32)$$

for $k = 1,2,3, \ldots$

So, the simplification of Equation (32) imply to the following equations:

$$w_{1}(t) = t - \frac{t^{2-\alpha}}{(2-\alpha)!} - \frac{t^3}{3!}, \qquad (33)$$

$$w_{2}(t) = t - t^3 + \frac{2t^5}{5!} - \frac{2t^{2-\alpha}}{(2-\alpha)!} + \frac{2t^{4-\alpha}}{(3-\alpha)!} + \frac{t^{3-2\alpha}}{(3-2\alpha)!}, \qquad (34)$$

$$w_{3}(t) = t - 2t^3 + \frac{2t^5}{3!} - \frac{17t^7}{315} - \frac{3t^{2-\alpha}}{(2-\alpha)!} + \frac{3t^{3-2\alpha}}{(3-2\alpha)!} - \frac{t^{4-3\alpha}}{(4-3\alpha)!} + \frac{6(3-\alpha)!t^{4-\alpha}}{(2-\alpha)!(4-\alpha)!} + \frac{6t^{4-\alpha}}{(4-\alpha)!} + \frac{2t^{4-\alpha}}{(3-\alpha)!} - \frac{2(5-\alpha)!t^{6-\alpha}}{3(2-\alpha)!(6-\alpha)!} - \frac{4(5-\alpha)!t^{6-\alpha}}{(3-\alpha)!(6-\alpha)!} - \frac{16t^{6-\alpha}}{(6-\alpha)!} - \frac{(4-2\alpha)!t^{5-2\alpha}}{[(2-\alpha)!]^2(5-2\alpha)!} - \frac{2(4-2\alpha)!t^{5-2\alpha}}{(3-2\alpha)!(5-2\alpha)!} - \frac{2(4-2\alpha)!t^{5-2\alpha}}{(3-\alpha)!(5-2\alpha)!}, \qquad (35)$$

Then, the approximated solution is given as follow:

$$w(t) = 4t - \frac{6t^{2-\alpha}}{(2-\alpha)!} - 10\frac{t^3}{3!} + \frac{4t^5}{5!} - \frac{2t^{2-\alpha}}{(2-\alpha)!} + \frac{2t^{4-\alpha}}{(3-\alpha)!} + \frac{t^{3-2\alpha}}{(3-2\alpha)!} - \frac{17t^7}{315} - \frac{3t^{2-\alpha}}{(2-\alpha)!} + \frac{3t^{3-2\alpha}}{(3-2\alpha)!} - \frac{t^{4-3\alpha}}{(4-3\alpha)!} + \frac{6(3-\alpha)!t^{4-\alpha}}{(2-\alpha)!(4-\alpha)!} + \frac{6t^{4-\alpha}}{(4-\alpha)!} + \frac{2t^{4-\alpha}}{(3-\alpha)!} - \frac{2(5-\alpha)!t^{6-\alpha}}{3(2-\alpha)!(6-\alpha)!} - \frac{4(5-\alpha)!t^{6-\alpha}}{(3-\alpha)!(6-\alpha)!} - \frac{16t^{6-\alpha}}{(6-\alpha)!} - \frac{(4-2\alpha)!t^{5-2\alpha}}{[(2-\alpha)!]^2(5-2\alpha)!} - \frac{2(4-2\alpha)!t^{5-2\alpha}}{(3-2\alpha)!(5-2\alpha)!} - \frac{2(4-2\alpha)!t^{5-2\alpha}}{(3-\alpha)!(5-2\alpha)!} + \ldots \qquad (36)$$

if $\alpha = 1$ we get the general solution as follow:

$$w(t) = tanh(t). \qquad (37)$$

Conclusion

In this paper, HPM has been modified for solving generalized semi-linear first-order FrDEs, it named MHPM. MHPM has been used for solving a class of FrDEs and the approximated solutions of this class have been derived using Maple software. The approximated results which are obtained by MHPM compared with the analytical solutions using MATLAB software. The comparison of the results show that the approximated solutions of the implementations in which plotted in Examples 1 and 2, a and b are agree well with the analytical solutions for the tested problems. Moreover, the proposed method is an efficient and high accurate method.

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