The Generalized Bivariate Jacobsthal and Jacobsthal Lucas Polynomial Sequences

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Abstract. From the definition of Fibonacci numbers (the first known special integer sequence), there are many studies on the integer sequences because of so many applications in science and art, and etc. For instance, the ratio of two consecutive elements of Fibonacci sequence is the golden ratio, is very important number almost every area of science and art. And the other integer sequence Jacobsthal numbers are met in computer science. It is well known that computers use conditional directives to change the flow of execution of a program. In addition to branch instructions, some microcontrollers use skip instructions which conditionally bypass the next instruction. This brings out being useful for one case out of the four possibilities on 2 bits, 3 cases on 3 bits, 5 cases on 4 bits, 11 cases on 5 bits, 21 cases on 6 bits, ..., which are exactly the Jacobsthal numbers. In this study, first of all we define and study the generalized bivariate Jacobsthal and generalized bivariate Jacobsthal Lucas polynomial sequences. Then the Binet formulae, some different types of generating functions, D' Ocagne, Catalan, Cassini properties and some interesting properties of these sequences are given. The sum of the square of elements of these sequences and some generalized sum formulae are obtained for the generalized bivariate Jacobsthal sequence and the bivariate Jacobsthal Lucas sequences. Finally, a divisibility property of the generalized bivariate Jacobsthal sequence is given.

Key words: bivariate polynomial sequences, Binet formula.

Introduction and Preliminaries

Jacobsthal and Jacobsthal Lucas numbers are given by the recurrence relations $j_n = j_{n-1}+2j_{n-2}$, $j_0 = 0$, $j_1 = 1$ and $c_n = c_{n-1}+2c_{n-2}$, $c_0 = 2$, $c_1 = 1$ for $n \geq 2$, respectively in works of Horadam (1996 40-54; 1997: 137-148) there are some generalizations of these integer sequences. For example, a generalization of Jacosthal sequences is given by Uygun (2015: 3467) as $j_n(s,t) = sj_{n-1}(s,t) + 2tj_{n-2}(s,t)$, $j_0(s,t) = 0$, $j_1(s,t) = 1$ and $c_n(s,t) = sc_{n-1}(s,t)+2tc_{n-2}(s,t)$, $c_0(s,t) = 2$, $c_1(s,t) = s$ for $n \geq 2$.

The bivariate Fibonacci $\{F_n(x,y)\}$ and Lucas $\{L_n(x,y)\}$ polynomials sequences are defined as by using the following recurrence relation (Koshy, 2001: 46; Köken and Bozkurt, 2008: 605-614)

$$F_n(x,y) = xF_{n-1}(x,y) + yF_{n-2}(x,y), \qquad (F_0(x,y) = 0,\ F_1(x,y) = 1)$$
$$L_n(x,y) = xL_{n-1}(x,y) + yL_{n-2}(x,y), \qquad (L_0(x,y) = 2,\ L_1(x,y) = x)$$

where $x \neq 0$, $y \neq 0$ and $x^2 + 4y \neq 0$. Some identities about the bivariate Fibonacci and Lucas polynomials are obtained by Catalani in (2004a, 2004b). And then the bivariate Pell and Pell Lucas polynomials are defined as by using the following recurrence relation

$$P_n(x,y) = 2xyP_{n-1}(x,y) + yP_{n-2}(x,y), \qquad (P_0(x,y) = 0,\ P_1(x,y) = 1)$$
$$Q_n(x,y) = 2xyQ_{n-1}(x,y) + yQ_{n-2}(x,y), \qquad (Q_0(x,y) = 2,\ Q_1(x,y) = 2xy)$$

where x≠ 0, $y \neq 0$ and $x^2y^2+8y\neq 0$. The Binet formula for these sequences are given in (Halici and Akyüz, 2010: 101-110) as

$$P_n(x,y)=\frac{\left(xy+\sqrt{x^2y^2+y}\right)^n-\left(xy-\sqrt{x^2y^2+y}\right)^n}{2\sqrt{x^2y^2+y}},$$

$$Q_n(x,y)=\left(xy+\sqrt{x^2y^2+y}\right)^n+\left(xy-\sqrt{x^2y^2+y}\right)^n.$$

In works by Halıcı and Akyüz (2010: 101-110) by using different matrices new sum formulae for bivariate Pell and Pell Lucas polynomials are obtained. In Tuglu et al. (2011: 10239) the authors study bivariate Fibonacci and Lucas p- polynomials sequences and give some properties of these sequences.

In this paper, we study the generalized bivariate Jacobsthal $\{j_n(x,y)\}$ and bivariate Jacobsthal Lucas $\{c_n(x,y)\}$ polynomial sequences in detail.

Results

Definition 1: Let $p(x,y)$, $q(x,y)$ be polynomials with real coefficients. For $n\geq 2$, the generalized bivariate Jacobsthal $\{j_n(x,y)\}$ and the generalized bivariate Jacobsthal Lucas $\{c_n(x,y)\}$ polynomials are described by using the following recurrence relations respectively (Civciv and Turkmen, 2008: 161-173; Nalli and Haukkanen: 2009: 3179).

$$j_n(x,y)=p(x,y)j_{n-1}(x,y)+2q(x,y)j_{n-2}(x,y),\ \ (j_0(x,y)=0,\ \ j_1(x,y)=1) \tag{1}$$

$$c_n(x,y)=p(x,y)c_{n-1}(x,y)+2q(x,y)c_{n-2}(x,y),\ \ (c_0(x,y)=2,\ \ c_1(x,y)=p(x,y)) \tag{2}$$

where and $p^2(x,y)+8q(x,y)>0$. The characteristic equation of recurrence relation (1) and (2)

$$r^2-p(x,y)r-2q(x,y)=0.$$

The roots of the characteristic equation are

$$\alpha(x,y)=\frac{p(x,y)+\sqrt{p^2(x,y)+8q(x,y)}}{2},\qquad \beta(x,y)=\frac{p(x,y)-\sqrt{p^2(x,y)+8q(x,y)}}{2}$$

with the following properties

$\alpha(x,y)+\ \beta(x,y)=p(x,y)$, $\alpha(x,y).\ \beta(x,y)=-2q(x,y)$,

$\alpha(x,y)-\ \beta(x,y)=\sqrt{p^2(x,y)+8q(x,y)}$.

The first some generalized bivariate Jacobsthal polynomials are $j_0(x,y)=0$, $j_1(x,y)=1$, $j_2(x,y)=p(x,y)$, $j_3(x,y)=p^2(x,y)+2q(x,y)$, $j_4(x,y)=p^3(x,y)+4p(x,y)q(x,y)$. The first some generalized bivariate Jacobsthal Lucas polynomials are $c_0(x,y)=2$, $c_1(x,y)=p(x,y)$, $c_2(x,y)=p^2(x,y)+4q(x,y)$, $c_3(x,y)=p^3(x,y)+6p(x,y)q(x,y)$. The Binet formulas for these sequences are

$$j_n(x,y)=\frac{(\alpha(x,y))^n-\left(\beta(x,y)\right)^n}{\alpha(x,y)-\beta(x,y)} \tag{3}$$

$$c_n(x,y)=(\alpha(x,y))^n+\left(\beta(x,y)\right)^n \tag{4}$$

The negative term for bivariate Jacobsthal and Jacobsthal Lucas polynomials are defined

$$j_{-n}(x,y)=\frac{-j_n(x,y)}{(-2q(x,y))^n}$$

$$c_{-n}(x,y)=\frac{c_n(x,y)}{(-2q(x,y))^n}$$

with the roots $1/\alpha$ and $1/\beta$. For the brievity, in the rest of the paper we use the notation j_n for $j_n(x,y)$ and c_n for $c_n(x,y)$, and α_n for $\alpha_n(x,y)$, β_n for $\beta_n(x,y)$.

Theorem 2: (Explicit closed form)

$$j_n=\sum_{k=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor}\binom{n-1-k}{k}\left(p(x,y)\right)^{n-1-2k}(2q(x,y))^k$$

and

$$c_n=\sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\binom{n-k}{k}\frac{n}{n-k}\left(p(x,y)\right)^{n-2k}(2q(x,y))^k$$

Proof: Induction on *n* provides the required proofs.

Theorem 3: (The Generating functions of generalized bivariate Jacobsthal and Jacobsthal Lucas polynomial sequences)

Let *i* any positive integer and $\alpha^i t<1$ and $\beta^i t<1$. Then the generating functions of these sequences for different values of i are obtained as

$$\sum_{n=0}^{\infty}j_{in}t^n=\frac{j_i t}{1-c_i t+(-2q(x,y))^i t^2},$$

$$\sum_{n=0}^{\infty}c_{in}t^n=\frac{2+t\left(\sqrt{p^2(x,y)+8q(x,y)}\right)j_i}{1-c_i t+(-2q(x,y))^i t^2}.$$

Proof: By using Binet formula for generalized bivariate Jacobsthal polynomial sequence, we get

$$\sum_{n=0}^{\infty}j_{in}t^n=\sum_{n=0}^{\infty}\frac{\alpha^{in}-\beta^{in}}{\alpha-\beta}t^n=\frac{1}{\alpha-\beta}\sum_{n=0}^{\infty}\left[\left(\alpha^i t\right)^n-\left(\beta^i t\right)^n\right]$$

$$=\frac{1}{\alpha-\beta}\left[\frac{1}{1-\alpha^i t}-\frac{1}{1-\beta^i t}\right]$$

$$=\frac{\left(\alpha^i-\beta^i\right)t}{(\alpha-\beta)\left(1-t\left(\alpha^i+\beta^i\right)+t^2(-2q(x,y))^i\right)}$$

$$=\frac{j_i t}{1-c_i t+(-2q(x,y))^i t^2}$$

Similarly

$$\sum_{n=0}^{\infty} c_{in} t^n = \sum_{n=0}^{\infty} \left(\alpha^{in} - \beta^{in}\right) t^n = \sum_{n=0}^{\infty} \left[\left(\alpha^i t\right)^n + \left(\beta^i t\right)^n\right]$$
$$= \left[\frac{1}{1-\alpha^i t} + \frac{1}{1-\beta^i t}\right]$$
$$= \frac{2-\left(\alpha^i+\beta^i\right)t}{\left(1-t\left(\alpha^i+\beta^i\right)+t^2(-2q(x,y))^i\right)} = \frac{2-c_i t}{1-c_i t+(-2q(x,y))^i t^2}$$

Theorem 4: The Exponential Generating Functions of generalized bivariate Jacobsthal and Jacobsthal Lucas Sequences

$$\sum_{n=0}^{\infty} j_n \frac{t^n}{n!} = \sum_{n=0}^{\infty} \frac{\alpha^n-\beta^n}{\alpha-\beta}\frac{t^n}{n!} = \frac{1}{\sqrt{p^2(x,y)+8q(x,y)}} \sum_{n=0}^{\infty} \frac{(\alpha t)^n - (\beta t)^n}{n!}$$
$$= \frac{1}{\sqrt{p^2(x,y)+8q(x,y)}}\left(e^{\alpha t} - e^{\beta t}\right)$$
$$\sum_{n=0}^{\infty} c_n \frac{t^n}{n!} = \left(e^{\alpha t} + e^{\beta t}\right)$$

Important Relationships

$$j_n c_{n=} j_{2n},$$
$$c_{n=} j_{n+1} + 2q(x,y) j_{n-1},$$
$$\left(p^2(x,y)+8q(x,y)\right) j_n = c_{n+1} + 2q(x,y) c_{n-1},$$
$$p(x,y) j_n + c_n = j_{n+1},$$
$$\left(p^2(x,y)+8q(x,y)\right) j_n + p(x,y) c_n = 2c_{n+1},$$
$$\sqrt{p^2(x,y)+8q(x,y)}\, j_n + c_{n=} 2\alpha^n,$$
$$\sqrt{p^2(x,y)+8q(x,y)}\, j_n - c_{n=} -2\beta^n,$$
$$c_{n+2}^2 + 2q(x,y) c_{n+1}^2 = c_{2n+4} + 2q(x,y) c_{2n+2},$$
$$j_{n+1}^2 + 2q(x,y) j_n^2 = \sqrt{p^2(x,y)+8q(x,y)}\, j_{2n+1},$$
$$c_{2n} = j_n^2\left(p^2(x,y)+8q(x,y)\right) + 2\left(-2q(x,y)\right)^n,$$
$$c_n^2 = c_{2n} + 2\left(-2q(x,y)\right)^n,$$
$$\left(p^2(x,y)+8q(x,y)\right) j_n^2 = c_{2n} - 2\left(-2q(x,y)\right)^n,$$
$$c_{3n} = c_n\left(c_{2n} - 2\left(-2q(x,y)\right)^n\right),$$
$$j_{3n} = j_n\left(c_{2n} + 2\left(-2q(x,y)\right)^n\right).$$

Proof: All of the proofs can be seen easily by using Binet formula or mathematical induction method.

Theorem 5: (Summation Formulas)

Let a,b are positive integers. For generalized bivariate Jacobsthal polynomial sequence, we get

$$\sum_{k=0}^{n-1} j_{ak+b} = \frac{j_b - j_{na+b} - \left(-2q(x,y)\right)^a j_b - a j_{(n+1)i} + \left(-2q(x,y)\right)^a j_{(n-1)a+b}}{1-c_a+\left(-2q(x,y)\right)^a}.$$

and for generalized bivariate Jacobsthal Lucas polynomial sequence, we get

$$\sum_{k=0}^{n-1} c_{ak+b} = \frac{\left(-2q(x,y)\right)^{a}\left[c_{a(n-1)+b} - c_{b-a}\right] - c_{an+b} + c_{b}}{1 - c_{a} + \left(-2q(x,y)\right)^{a}}.$$

Theorem 6: (D'ocagne's property)

Let $n \geq m$ and $n, m \in \mathbb{Z}^{+}$. For generalized bivariate Jacobsthal polynomial sequence, we have

$$j_{m+1}j_{n} - j_{m}j_{n+1} = \left(-2q(x,y)\right)^{m} j_{n-m}.$$

Let $m \geq n$ and $n, m \in \mathbb{Z}^{+}$. For generalized bivariate Jacobsthal Lucas polynomial sequence, we have

$$c_{m+1}c_{n} - c_{m}c_{n+1} = \sqrt{p^{2}(x,y) + 8q(x,y)}\left(-2q(x,y)\right)^{n} c_{m-n}.$$

Proof: By (3) and (4), we have

$$\begin{aligned} j_{m+1}j_{n} - j_{m}j_{n+1} &= \frac{\propto^{m+1} - \beta^{m+1}}{\alpha - \beta}\frac{\propto^{n} - \beta^{n}}{\alpha - \beta} - \frac{\propto^{m} - \beta^{m}}{\alpha - \beta}\frac{\propto^{n+1} - \beta^{n+1}}{\alpha - \beta} \\ &= \frac{1}{(\alpha - \beta)^{2}}\left[-\propto^{m+1}\beta^{n} - \beta^{m+1}\propto^{n} + \propto^{m}\beta^{n+1} + \propto^{n+1}\beta^{m}\right] \\ &= \frac{1}{(\alpha - \beta)^{2}}\left[\propto^{n}\beta^{m}(\alpha - \beta) - \propto^{m}\beta^{n}(\alpha - \beta)\right] \\ &= \frac{1}{\alpha - \beta}\left[(\propto - \beta)^{m}\left(\propto^{n-m} - \beta^{n-m}\right)\right] \\ &= \frac{1}{\alpha - \beta}\left[\left(-2q(x,y)\right)^{m}\left(\propto^{n-m} - \beta^{n-m}\right)\right]. \end{aligned}$$

It can be proved for generalized bivariate Jacobsthal Lucas numbers by using the same method for generalized bivariate Jacobsthal numbers (Horadam and Mahon, 1985: 7-20; Uslu and Uygun, 2013: 13-22).

Theorem 7: (Catalan's property)

Assume that $n, r \in \mathbb{Z}^{+}$. For generalized bivariate Jacobsthal polynomial sequence, we have

$$j_{n+r}j_{n-r} - j_{n}^{2} = -\left(-2q(x,y)\right)^{n-r} j_{r}^{2}$$

and for generalized bivariate Jacobsthal Lucas polynomial sequence,

$$c_{n+r}c_{n-r} - c_{n}^{2} = \left(-2q(x,y)\right)^{n-r} j_{r}^{2}\left(p^{2}(x,y) + 8q(x,y)\right).$$

Theorem 8: (Cassini's property or Simpson property)

For $n \in \mathbb{Z}^{+}$, it is obtained that

$$j_{n+1}j_{n-1} - j_{n}^{2} = -\left(-2q(x,y)\right)^{n-1},$$

and

$$c_{n+1}c_{n-1} - c_{n}^{2} = \left(-2q(x,y)\right)^{n-1}\left(p^{2}(x,y) + 8q(x,y)\right).$$

Theorem 9: For generalized bivariate Jacobsthal Lucas polynomial sequence, the following results are satisfied

$$j_{4n+p}-(2q(x,y))^{2n}j_p=j_{2n}c_{2n+p},$$

$$j_{4n+p}+(2q(x,y))^{2n}j_p=c_{2n}j_{2n+p},$$

$$j_{3n+p}-(-2q(x,y))^{n}j_{n+p}=j_nc_{2n+p},$$

$$j_{3n+p}+(-2q(x,y))^{n}j_{n+p}=c_nj_{2n+p},$$

where $n\geq 1$, $p\geq 0$.

It can be proved by using Binet formulas as the following theorem.

Theorem 10: For generalized bivariate Jacobsthal Lucas polynomial sequence, the following results are satisfied

$$c_{4n+p}-(2q(x,y))^{2n}c_p=(p^2(x,y)+8q(x,y))j_{2n}j_{2n+p},$$

$$c_{4n+p}+(2q(x,y))^{2n}c_p=c_{2n}c_{2n+p},$$

$$c_{3n+p}-(-2q(x,y))^{n}c_{n+p}=(p^2(x,y)+8q(x,y))j_nj_{2n+p},$$

$$c_{3n+p}+(-2q(x,y))^{n}c_{n+p}=c_nc_{2n+p},$$

where $n\geq 1$, $p\geq 0$.

Theorem 11: Let $n\geq 0$ any integer. Then generating function with negative indices are obtained as

$$\sum_{k=0}^{n} j_kt^{-k}=\frac{-1}{t^n(t^2-p(x,y)t-2q(x,y))}\left[-t^{n+1}-tj_{n+1}+2tj_n\right]$$

$$\sum_{k=0}^{n} c_kt^{-k}=\frac{-1}{t^n(t^2-p(x,y)t-2q(x,y))}\left[tc_{n+1}+2q(x,y)c_n\right]+\frac{2t^2-p(x,y)t}{(t^2-p(x,y)t-2q(x,y))}$$

Proof: By using expansion of geometric series, it is calculated as

$$\sum_{k=0}^{n} j_k t^{-k} = \frac{1}{\alpha-\beta}\left[\frac{1-\left(\frac{\alpha}{t}\right)^{n+1}}{1-\frac{\alpha}{t}} - \frac{1-\left(\frac{\beta}{t}\right)^{n+1}}{1-\frac{\beta}{t}}\right]$$

$$= \frac{1}{(\alpha-\beta)t^{n}}\left[\frac{t^{n+1}-\alpha^{n+1}}{t-\alpha} - \frac{t^{n+1}-\beta^{n+1}}{t-\beta}\right]$$

$$= \frac{-1}{(\alpha-\beta)t^{n}}\left[\frac{-t^{n+1}(\alpha-\beta)+t(\alpha^{n+1}-\beta^{n+1})+2t(\alpha^{n}-\beta^{n})}{t^{2}-p(x,y)t-2q(x,y)}\right]$$

$$= \frac{-1}{t^{n}}\left[\frac{-t^{n+1}+tj_{n+1}+2tj_{n}}{t^{2}-p(x,y)t-2q(x,y)}\right].$$

$$\sum_{k=0}^{n} c_k t^{-k} = \frac{1}{\alpha-\beta}\left[\frac{1-\left(\frac{\alpha}{t}\right)^{n+1}}{1-\frac{\alpha}{t}} + \frac{1-\left(\frac{\beta}{t}\right)^{n+1}}{1-\frac{\beta}{t}}\right]$$

$$= \frac{1}{t^{n}}\left[\frac{t^{n+1}-\alpha^{n+1}}{t-\alpha} + \frac{t^{n+1}-\beta^{n+1}}{t-\beta}\right]$$

$$= \frac{-1}{t^{n}}\left[\frac{-2t^{n+2}+t^{n+1}(\alpha+\beta)+t(\alpha^{n+1}+\beta^{n+1})+2q(x,y)(\alpha^{n}+\beta^{n})}{t^{2}-p(x,y)t-2q(x,y)}\right]$$

$$= \frac{-1}{t^{n}}\left[\frac{tc_{n+1}+2q(x,y)c_{n}}{t^{2}-p(x,y)t-2q(x,y)}\right] + \frac{2t^{2}-p(x,y)}{t^{2}-p(x,y)t-2q(x,y)}.$$

Conclusion 12: If we take $n \to \infty$ in the above theorem, we get

$$\sum_{i=0}^{\infty} j_i t^{-i} = \frac{t}{(t^{2}-p(x,y)t-2y)}.$$

Conclusion 13: If we take $n \to \infty$ in the above theorem, we get

$$\sum_{i=0}^{\infty} c_i t^{-i} = \frac{2t^{2}-p(x,y)t}{t^{2}-p(x,y)t-2q(x,y)}.$$

Theorem 14: Let r is any positive integer and $\left|\alpha^{k}\beta^{r-k}t\right| < 1$, then

$$\sum_{i=0}^{\infty} j_i{}^r t^i = \sum_{k=0}^{r} \binom{r}{k} \frac{(-1)^{r-k}}{(\sqrt{(p^2(x,y)+8q(x,y))})^r} \frac{1}{1-\alpha^k(-\beta)^{r-k} t}.$$

Proof: By using geometric series and Binet formula, we have

$$\sum_{i=0}^{\infty} j_i{}^r t^i = \sum_{i=0}^{\infty} \sum_{k=0}^{r} \binom{r}{k} \left(\frac{\alpha^i}{\alpha-\beta}\right)^k \left(\frac{-\beta^i}{\alpha-\beta}\right)^{r-k} t^i$$

$$= \sum_{k=0}^{r} \binom{r}{k} \left(\frac{1}{\alpha-\beta}\right)^r \sum_{i=0}^{\infty} \left(\alpha^k (-\beta)^{r-k} t\right)^i$$

$$= \sum_{k=0}^{r} \binom{r}{k} \frac{1}{(\sqrt{(p^2(x,y)+8q(x,y))})^r} \frac{1}{1-\alpha^k (-\beta)^{r-k} t}.$$

Theorem 15: Let r is any positive integer and $\left|\alpha^k \beta^{r-k} t\right| < 1$, then

$$\sum_{i=0}^{\infty} c_i{}^r t^i = \sum_{k=0}^{r} \binom{r}{k} \frac{1}{1-\alpha^k \beta^{r-k} t}.$$

Proof: By using geometric series and Binet formula, it is calculated as

$$\sum_{i=0}^{\infty} c_i{}^r t^i = \sum_{i=0}^{\infty} \sum_{k=0}^{r} \binom{r}{k} \left(\alpha^i\right)^k \left(-\beta^i\right)^{r-k} t^i$$

$$= \sum_{k=0}^{r} \binom{r}{k} \sum_{i=0}^{\infty} \left(\alpha^k \beta^{r-k} t\right)^i$$

$$= \sum_{k=0}^{r} \binom{r}{k} \frac{1}{1-\alpha^k \beta^{r-k} t}.$$

Theorem 16: By this theorem new relations between the roots α, β and generalized bivariate Jacobsthal and Jacobsthal Lucas polynomial sequences are demonstrated

$$\alpha^n = \alpha j_n + 2q(x,y) j_{n-1},$$

$$\beta^n = \beta j_n + 2q(x,y) j_{n-1},$$

$$\sqrt{p^2(x,y)+8q(x,y)}\,\alpha^n = \alpha c_n + 2q(x,y) c_{n-1},$$

$$-\sqrt{p^2(x,y)+8q(x,y)}\,\beta^n = \beta c_n + 2q(x,y) c_{n-1}.$$

Proof: The proof is made by using Binet formula and the product of the roots:

$$I = \alpha j_n + 2q(x,y) j_{n-1}$$
$$I = \beta \frac{\alpha^n - \beta^n}{\alpha - \beta} + 2q(x,y)\frac{\alpha^{n-1} - \beta^{n-1}}{\alpha - \beta}$$
$$= \frac{1}{\alpha - \beta}\left[\beta\left(\alpha^n - \beta^n\right) + 2q(x,y)\left(\alpha^{n-1} - \beta^{n-1}\right)\right]$$
$$= \frac{1}{\alpha - \beta}\left[-2q(x,y)\alpha^{n-1} - \beta^{n+1} + 2q(x,y)\alpha^{n-1} - 2q(x,y)\beta^{n-1}\right]$$
$$= \frac{1}{\alpha - \beta}\left[-\beta^{n-1}(\beta^2 + 2q(x,y))\right] = \beta^n$$

Similarly

$$I = \alpha c_n + 2q(x,y) c_{n-1},$$
$$I = \alpha\left(\alpha^n + \beta^n\right) + 2q(x,y)\left(\alpha^{n-1} + \beta^{n-1}\right)$$
$$= \alpha^{n+1} - 2q(x,y)\beta^{n-1} + 2q(x,y)\beta^{n-1} + 2q(x,y)\alpha^{n-1}$$
$$= \alpha^{n-1}(\alpha^2 + 2q(x,y))$$
$$= \alpha^n(\alpha - \beta) = \alpha^n\sqrt{p^2(x,y) + 8q(x,y)}$$

Other proofs can be done by using the same way.

Theorem 17: The square of elements of generalized bivariate Jacobsthal sequence is obtained by the following:

$$\sum_{i=0}^{n-1} j_i^2 = \frac{1}{p^2(x,y) + 8q(x,y)}\left(\frac{4q(x,y)^2 c_{2n-2} - c_{2n} - c_2 + 2}{1 + 4q(x,y)^2 - c_2} + 2\frac{(-2q(x,y))^n - 1}{2q(x,y) + 1}\right)$$

Proof: By the definition of Binet formulas, we have

$$\sum_{i=0}^{n-1} j_i^2 = \frac{1}{p^2(x,y) + 8q(x,y)}\sum_{i=0}^{n-1}\left(\alpha^{2i} + \beta^{2i} - 2(-2q(x,y))^i\right)$$
$$= \frac{1}{p^2(x,y) + 8q(x,y)}\left(\frac{\alpha^{2n} - 1}{\alpha^2 - 1} + \frac{\beta^{2n} - 1}{\beta^2 - 1} + 2\frac{(-2q(x,y))^n - 1}{2q(x,y) + 1}\right)$$
$$= \frac{1}{p^2(x,y) + 8q(x,y)}\left(\frac{4q(x,y)^2 c_{2n-2} - c_{2n} - c_2 + 2}{1 + 4q(x,y)^2 - c_2} + 2\frac{(-2q(x,y))^n - 1}{2q(x,y) + 1}\right)$$

Theorem 18: The square of elements of generalized bivariate Jacobsthal Lucas sequence is obtained by the following:

$$\sum_{i=0}^{n-1} c_i^2 = \frac{4q(x,y)^2 c_{2n-2} - c_{2n} - c_2 + 2}{1 + 4q(x,y)^2 - c_2} - 2\frac{(-2q(x,y))^n - 1}{2q(x,y) + 1}.$$

Proof: By the definition of Binet formulas, we have

$$\sum_{i=0}^{n-1} c_i^{\,2} = \sum_{i=0}^{n-1}\left(\alpha^{2i}+\beta^{2i}+2(-2q(x,y))^i\right)$$
$$=\frac{\alpha^{2n}-1}{\alpha^2-1}+\frac{\beta^{2n}-1}{\beta^2-1}-2\frac{(-2q(x,y))^n-1}{2q(x,y)+1}$$
$$=\frac{4q(x,y)^2c_{2n-2}-c_{2n}-c_2+2}{1+4q(x,y)^2-c_2}-2\frac{(-2q(x,y))^n-1}{2q(x,y)+1}$$

Theorem 19: By this theorem we can see another sum property, equals to $2n$. th element of generalized bivariate Jacobsthal, generalized bivariate Jacobsthal Lucas sequence respectively,

$$\sum_{i=0}^{n}\binom{n}{i}(2q(x,y))^{n-i}(p(x,y))^i j_i = j_{2n},$$

$$\sum_{i=0}^{n}\binom{n}{i}(2q(x,y))^{n-i}(p(x,y))^i c_i = c_{2n}.$$

Proof:

$$\sum_{i=0}^{n}\binom{n}{i}(2q(x,y))^{n-i}(p(x,y))^i j_i = \frac{1}{\alpha-\beta}\left(\begin{array}{l}\sum_{i=0}^{n}\binom{n}{i}(2q(x,y))^{n-i}(\alpha p(x,y))^i\\ -\sum_{i=0}^{n}\binom{n}{i}(2q(x,y))^{n-i}(\beta p(x,y))^i\end{array}\right)$$
$$=\frac{1}{\alpha-\beta}\left((2q(x,y)+\alpha p(x,y))^n+(2q(x,y)+\beta p(x,y))^n\right)$$
$$=j_{2n}.$$

$$\sum_{i=0}^{n}\binom{n}{i}(2q(x,y))^{n-i}(p(x,y))^i c_i = \left(\begin{array}{l}\sum_{i=0}^{n}\binom{n}{i}(2q(x,y))^{n-i}(\alpha p(x,y))^i\\ +\sum_{i=0}^{n}\binom{n}{i}(2q(x,y))^{n-i}(\beta p(x,y))^i\end{array}\right)$$
$$=\left((2q(x,y)+\alpha p(x,y))^n+(2q(x,y)+\beta p(x,y))^n\right)=c_{2n}.$$

Divisibility Properties of Bivariate Jacobsthal Polynomial Sequence

Lemma 20: For m, n positive integers

$$j_{m+n+1} = j_{m+1}j_{n+1}+2q(x,y)j_m j_n.$$

Theorem 21: Let n≥2 positive integers,

$$j_m / j_n \Leftrightarrow m / n.$$

Proof: $\Leftarrow$Assume that *m/n*, then there exists an integer *k* such that *n=km*. We want to show j_m / j_n. We use induction method. For *k=1*, it's easily seen that j_m / j_m. Suppose that j_m / j_{km}. For *k=n+1*, from the above Lemma

$$j_{(k+1)m} = j_{km} j_{m+1} + 2q(x,y) j_{km-1} j_m.$$

Since j_m / j_{km} then it's easily seen that $j_m / j_{(k+1)m}$

$\Rightarrow$ Let j_m / j_n.} and *m* $\nmid$*n*. So there exist integers *q, r* with *0<r<m*, such that *n=mq+r*. From the above theorem

$$j_n = j_{mq+r} = j_{mq+1} j_r + 2q(x,y) j_{mq} j_{r-1}.$$

Since j_m / j_{qm} then $j_m / j_{qm+1} j_r$. Since $(j_n, j_{n+1}) = y$ then j_m / j_r. This is impossible because of lower degree of j_r than j_m in *x* (Uygun, 2018a, 2018 b). So we have *r=0* and *m/n.*

Conclusion

In this paper we have obtained a lot of interesting properties, are satisfied by generalized bivariate Jacobsthal and Jacobsthal Lucas polynomials. We give some generating functions, explicit formulas, different sum properties, divisibility properties, partial derivatives etc.

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