Rectification of a Circle Using Modified Kochański's and Pupovac's Method

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Abstract. The problem of the exact rectification of a circle cannot be solved by classical geometry. Many approximate methods have been developed. Such an elegant one is Kochański's construction. Another one is the less known Pupovac's method. Variants of these constructions are presented here.
Key words: Kochański's method, Pupovac's method, rectification, circle.

Introduction

The exact rectification of a circle, i.e., a construction of a linear segment exactly of the length of the circumference of a given circle, has been proven to be an impossible geometrical construction using only a straightedge and compass. The formula to determine the exact length of the circumference depends on the number π. A German mathematician von Lindemann in 1882 (Von Lindemann, 1882: 213-225) has shown that the mathematical constant π is not an algebraic number. Thus, the rectification is an impossible construction to realize, as a consequence that the π is transcendental number. Of course, the length of the circumference of a circle may have any value. For example, for a circle of the diameter equal to 1/π, the length of the circumference is 1.

In the Ancient World, Sumerian (later Babylon) and Egyptian civilizations, coexisted two totally different approaches to determine the area of a circle. A Sumerian civilization in Mesopotamia used the approach based directly on the circumference of a circle. For the estimated length of the circumference, say C, the area A of the circle was calculated as $A = C * C/12$. They used 5 rather than our 12, as 5/60=12. We do not know exactly how C was determined. It was probably based on a regular hexagon and the value of C was assumed to be 6 times the length of the radius of a circle. Even for an exact value of C (=2 π r), the formula used for the area implies that the number π is 3.

Egyptians used a different approach. The method presented on an ancient Egyptian papyrus and recorded by the scribe Ahmes is based on the diameter of a circle. This approach is much closer to our modern rule "area of the circle is π multiplied by its radius squared". To obtain the area of a circle the scribe Ahmes used the following rule (Problem 50 (Szyszkowicz, 2019: 156-158): "The area of a circle with diameter d is the area of a square with side 8d/9."

We know very well Archimedes' approach used to estimate the value of the number π. Archimedes probably knew both methods (Babylonian and Egyptian) to calculate the area of a circle. In his work, he was able to obtain a good approximation of the number π. He estimated its value by approximating the length of the circumference of the circle with diameter equal to 1. Archimedes also demonstrated that the area of the circle is the same as the area of the right-angle triangle, whose base equals to the length of the circumference and its height is the radius of the circle. Finally, the area of the circle is expressed as "π multiplied by the radius squared". Archimedes merged the "circumference (Babylon)" method with the "diameter (Egypt)" method. It is well described by the following formulae: $A = r * \frac{C}{2} = r * 2 * \pi * \frac{r}{2} = \pi r^2$. This formula transfers the area of the triangle to the area of a square. It also shows that the squaring of the circle is equivalent to the rectification of its circumference. This presentation uses two rectification methods, one is known another is forgotten, and provides their modifications.

Kochański's method

There is a number of rectification constructions that are realised in a simple way and provide a pretty good approximation; among them is beautiful Kochański's geometrical construction, which gives 0.002% accuracy (Fukś, 2011). Kochański's method is represented in Fig. 1. Consider for simplicity the unit circle (radius is 1). Kochański's construction needs only one opening of the compass. The length of the segment AB, (|AB|) is 3.141533(3)… It is easy to demonstrate this property as we known two other sides of this right-angle triangle (ABC):|AC|=2 and by the construction $|BC| = 3 - \tan(30^o) = 3 - \frac{\sqrt{3}}{3}$. From Pythagoras' theorem $|AB| = \sqrt{|AC|^2 + |BC|^2}$ $= \sqrt{4 + (3 - \frac{\sqrt{3}}{3})^2} = \sqrt{\frac{40}{3} - 2\sqrt{3}} = 3.141533(3) \cdots \approx \pi$.

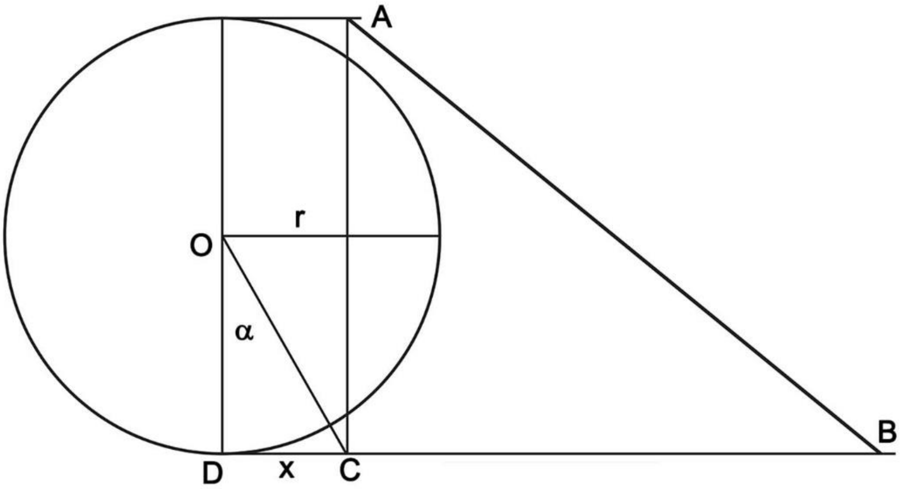

Fig. 1. Kochański's construction (α=30^o)

Kochański's approximate ruler-and-compass construction provides four digits of accuracy of the number π. In his construction |DB|=3 and the angle α is 30^o. Adam Adamandy Kochański was a Polish Jesuit and a mathematician (Kochański, 1685). He published his rectification method in the year 1685 (Ley, 1965: 79-81).

Pupovac's method

In the year 1910 Peter Pupovac, an Austrian civil servant, presented another a similar and simple geometrical construction (Ley, 1965: 79-81). His method is described by the following steps. The diameter of the circle (d) is divided into five equal parts. We can apply Thales' theorem on proportions to execute this division of the diameter (see Fig. 2). One part is added to extend the diameter and to obtain point B. From this point a perpendicular line to the segment AB is constructed. Now three parts are checked off on the vertical line to determine point C. In this way we constructed the right-angle triangle ABC. The triangle has the property that the sum of its sides approximates the length of the circumference of the circle. We can easily verify this. Two of its sides, AB and BC, have the length $\frac{6}{5}d$ and $\frac{3}{5}d$, respectively. The length of the hypotenuse (|AC|) is calculated according to Pythagoras' rule $(|AB|^2 + |BC|^2 = |AC|^2)$ and it is $|AC| = \sqrt{\left(\frac{6}{5}\right)^2 + \left(\frac{3}{5}\right)^2} = \frac{3}{5}\sqrt{5}$. The total sum of the sides of triangle ABC measures $1.8 * \sqrt{5}$ which is 3.14164… multiplied by the diameter (d) of the circle. Pupovac's method gives three digits of accuracy for the number π. As we see, this construction gives relatively a good approximation of the circumference of the circle.

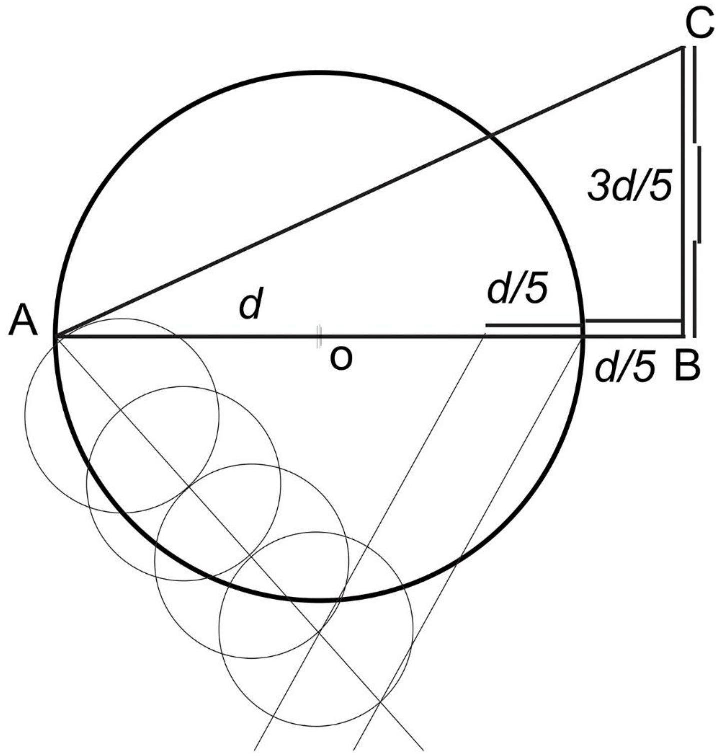

Fig. 2. Pupovac's construction

Adding the sides AB, BC, and AC together on a straight line, and taking one quarter (halved twice) of the resulting segment we construct the approximation to $\frac{\pi}{4}d = \frac{\pi}{2}r$, where r is a radius of the circle (d=2r). We can square the rectangle with this side and with another side equals to d, i.e. the rectangle described by its sides as $(\frac{\pi}{2}r\ x\ d)$, to obtain the side of the square. Its area is an approximation of the area of the circle. The constructed square realizes the approximate squaring of the circle.

Modified Kochański's method

Kochański's construction is performed using only classical geometrical tools, a compass and straightedge. We may introduce and build another geometrical device to perform this construction. This device allows to realize the construction automatically. In this case it is a special tool, a square set which allows to realize squaring the circle in the spirit of Kochański's approach. Rather than keep the value of the angle α (in Kochański's method it is 30 degrees) as fixed, we allow that this angle will be changed. The question here is about the true value of the angle. We take the tangent of this angle. The construction uses the triangle ABC. Consider the variable x (x=tan(α)) which can be determined from the following equation (see Fig. 1)

$$4r^2 + (3 - x)^2 r^2 = (\pi r)^2$$

It is relatively easy to solve. The solution does not depend on the radius of the circle. The equation has two solutions.

$$x_{1,2} = 3 \mp \sqrt{\pi^2 - 4}$$

We mainly focus on the first root of the equation $x_1 = 0.577273354030761$ ..., where the corresponding angle α is 29.9966947044324… . As we see Kochański's value 30 degrees for the angle is very close to the exact value.

We approximate the tangent of the angle by continued fractions. For the first 9 terms we obtained the following values $\tan(\alpha) = x_1 = 0.577273354030761 \ldots \approx [0;\ 1\ 1\ 2\ 1\ 2\ 1\ 3\ 2]$. Table 1 summarizes the results. Constructing a square set with the sides in the proportion 56:97 we obtain almost the same accuracy as with Kochański's construction. The next proportion 127:220 results with the approximating of the number π as 3.141593136931727. In this situation we have five correct digits.

Table 1. Modified Kochański's method with root x_1. A sequence of the approximations to the tangent based on continued fractions

Numerator	Denominator	Tan (α)	Estimated π value
0	1	0.00000000000000	3.60555127546399
1	1	1.00000000000000	2.82842712474619
1	2	0.50000000000000	3.20156211871642
3	5	0.60000000000000	3.12409987036266
4	7	0.571428571428571	3.14610220792177
11	19	0.578947368421053	3.140301871616063
15	26	0.576923076923077	3.141862787447589
56	97	0.577319587628866	3.141556999401216
127	220	0.577272727272727	3.141593136931727

By an analogy, for the second solution we have $\tan(\alpha) = x_2 = 5.42272664596924 \ldots \approx [5; 2\ 2\ 1\ 2\ 1\ 3\ 2\ 32]$. In this case, the corresponding angle is 79.5515224436244… degrees. Table 2 summarizes the results. Here we have the following symmetry related to the length of the estimated segment corresponding to two roots, $3\text{-}x_1 = x_2\text{-}3$.

Table 2. Modified Kochański's method with root x_2. A sequence of the approximations to the tangent based on continued fractions

Numerator	Denominator	tan(α)	Estimated π value
5	1	5.00000000000000	2.82842712474619
11	2	5.50000000000000	3.20156211871642
27	5	5.40000000000000	3.12409987036266
38	7	5.42857142857143	3.14610222079218
103	19	5.42105263157895	3.14030187161606
141	26	5.42307692307692	3.14186278744759
526	97	5.42268041237113	3.14155699940122
1193	220	5.42272727272727	3.14159313693173
38702	7137	5.42272663584139	3.14159264577942

Modified Pupovac's method

We can define Pupovac's method slightly differently. Again, the right-angle triangle is constructed for the same purpose. The sum of the lengths of its sides approximates the length of the circumference of a given circle. One side of the triangle is the diameter. The given conditions (a sum of its sides is π*d) allow to determine the triangle with such predefined property. Consider the triangle T as shown on upper panel of Fig. 3. Define its two sides as d, xd, and by a consequence of Pythagoras' rule its hypotenuse is given as $\sqrt{d^2 + x^2d^2}$. The following equation specifies the assumed condition: the sum of the sides approximate the length of the circumference of the circle.

$$d + xd + \sqrt{d^2 + x^2d^2} = \pi d$$

We simplify this equation by the variable d. The construction doesn't dependent on the circle. The value x of the tangent of the angle α is now given as a root of the equation

$$1 + x + \sqrt{1 + x^2} = \pi$$

The equation has one solution of the form

$$x = \frac{\pi(\pi - 2)}{2(\pi - 1)} \approx 0.837325223332767 \dots .$$

In a similar way as for the modified Kochański's approach we approximate the value of the tangent by continued fractions. Its terms were determined as [0; 1 5 6 1 3 1 4 7]. The corresponding angle is 39.9402877443627, x=tan(α) and $atan(x) * \frac{180}{pi} = 39.9402877443627$, where $\tan(a) = x = 0.837325223332767$.

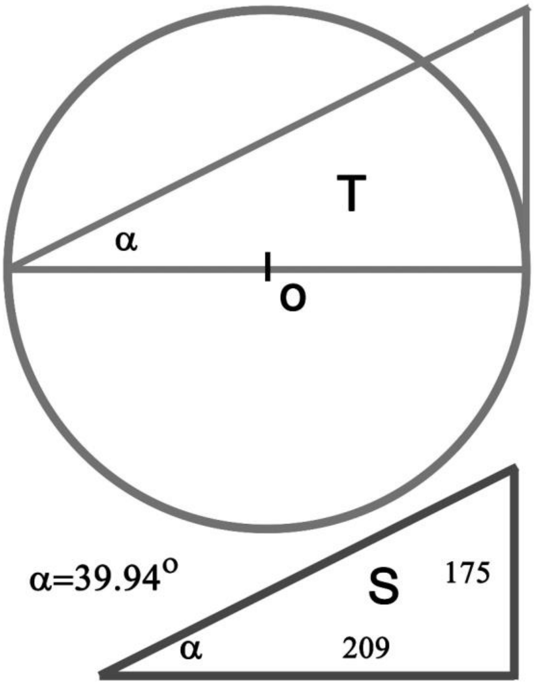

Fig. 3. Modified Pupovac's construction with a set square

Table 3. Modified Pupovac's method. A sequence of the approximations to the tangent based on continued fractions

Numerator	Denominator	tan(α)	Estimated π value
0	1	0	2
1	1	1.00000000000000	3.41421356237309
5	6	0.833333333333333	3.135041612651110
31	37	0.837837837837838	3.142434420124018
36	43	0.837209302325581	3.141402315610228
139	166	0.837349397590361	3.141632347584319
175	209	0.837320574162679	3.141585019708949
839	1002	0.837325349301397	3.141592860428891

6048	7223	0.837325211131109	3.141592633554807

According to the results presented in Table 3, a good choice is to use the proportion 175:209. It is relatively easy to build the right-angle triangle (a set square) with the sides in this proportion. Using the set square, we can automate the squaring of a circle with accuracy for the number π as 3.141585019708949. Fig. 3. (low panel) represents the set square S which allows to perform this process. It is the set square which generates the solution with four digits of accuracy.

Discussion and Conclusion

In this work, the presented constructions are used to satisfy the conditions related to the approximate rectification of the circumference. The condition for the squaring of the circle can be defined in a similar way. For example, in Pupovac's modified method the condition can be specified to result in a triangle which has the same area as the circle (Szyszkowicz, 2016: 57-60; 8-10; Szyszkowicz, 2017a: 301-302; Szyszkowicz, 2017b: 50-53).

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