# A Compression Study of Multistep Iterative Methods for Solving Ordinary Differential Equations

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**Abstract.** The objectives of this paper are to introduce the definition of generalized non expansive mapping to analyse the new iterative methods for solving initial value problems (IVPs) of ordinary differential equations (ODEs) and to compare the approximated results by using proposed method. The extended fixed point theorem in complete metric space has been introduced. The convergence analysis of the fixed point method in which generalizes non expansive type mapping in suitable space has been discussed. The numerical solution of the implementation has been studied by comparing the new iterative method with classical methods; Euler, modified Euler and the successive over- relaxation (SOR) methods by using MATLAB. The proposed method has more accuracy and efficiency and less of time complexity than the classical method. The approximated solution is agreeing well with analytical solution for the tested problem.

**Key words**: ODE, IVP, Euler method, fixed point, strong convergence, non-expansive mapping, iterative methods, differential equation.

## Introduction

Differential equations (DEs) play significant role in mathematical models and various of applications of mathematics for examples, physical phenomena, vibrations, chemical reactions, motion of objects, engineering analysis, branches of sciences, Fluid and heat flow, nuclear reactions, computer studies and bending and cracking of materials. Many applications of DEs particularly ODEs or partial differential equations (PDEs) of different orders used to construct some mathematical modelling in real-life problems (Mechee et al., 2014: 663-674). Many mathematical models in engineering and physics are expressed using DEs. Many researchers developed families of analytical or numerical methods for solving DEs. Various classes of numerical methods, especially iterative methods have been derived or developed for solving different types of DEs (Mechee and Rajihy, 2017: 2923-2949). Nowadays, there are more interest in needing of iterative methods using fixed point theory on normed linear spaces, Banach spaces and Hilbert spaces, respectively. Fixed point theory has been proposed as one of the most powerful and substantial theoretical tools of mathematics. In this paper, we have introduced the definition of generalized non expansive mapping and the analysis of new iterative methods for solving ODEs with initial condition and to compare the approximated results of proposed method with classical methods. Also, we have introduced the extended fixed point theorem in complete metric space. However, we conclude the numerical results show that the proposed method is more efficient and accurate than classical methods with less time complexity.

## Preliminary

In this section, we have defined a generalized non expansive mapping and we proved the extended fixed point theorem. Also, we introduced new types of iterative methods for the approximate solution of DEs. It is well known the Banach contraction theorem using Picard iteration algorithms for approximation of fixed point. The convergence of the iterative method is studied by many authors, see Maibed (2011; 2013; 2018); Hadi and Abd (2017), Abed and Maibed (2015).

Let $X$ be normed space, $T: C \longrightarrow C$ is $\text{non} - \text{expansive}$ mapping. In 1998, Xu (Xu, 1998) introduced the Ishikawa iteration as follows:

i-Let C be a nonempty convex subset of $X$, and $T: C \longrightarrow C$ a mapping. For any given $x_0 \in C$ the sequence $<x_n>$ defined by

$x_{n+1} = \alpha_n x_n + \beta_n T y_n + \gamma_n u_n$ and $y_n = \alpha'_n x_n + \beta'_n T y_n + \gamma'_n v_n$ , $n \geq 0$

is called sequence of Ishikawa iteration with errors.

Here $< u_n >$, $< v_n >$ are two bounded sequences in $C$, and the six sequences $<\alpha_n>$, $<\beta_n>$, $<\gamma_n>$, $< \alpha'_n >$, $< \beta'_n >$ and $< \gamma'_n >$ are in the interval [0,1] which satisfying

$\alpha_n + \beta_n + \gamma_n = \alpha'_n + \beta'_n + \gamma'_n = 1$, for all $n \geq 0$.

**ii-** In case, $\beta'_n = \gamma'_n$ for all $n \geq 0$ in (i), then $< x_n >$ defined by

$x_0 \in C$, $x_{n+1} = \alpha_n x_n + \beta_n T x_n + \gamma_n u_n$, $n \geq 0$

is called Mann iteration sequence with errors.

*Definition* (given by Zeidler, 1986)

1- A mapping $R : C \rightarrow M$ is called Lipchitz continuous with constant $\alpha > 0$
$|| Ra - Rb|| \leq \alpha|| a - b||$ , for any $a, b \in C$

2- I f $\alpha \in (0,1) \Rightarrow$ R is called contraction mapping .

3- If $\alpha = 1 \Rightarrow$ R is called non expensive mapping.

*Definition 2*

Let $M$ be a normed space. A mapping $R: M \longrightarrow M$ is called generalized non expansive if there exists $a, b \in [0,1]$ such that:

$d\big(R(x), R(y)\big) \leq (1 - a)d(x, y) + b\big|d\big(x, R\big(f(x)\big) - d(y, R(y)\big)\big|$,

for all $x, y \in M$.

*Example 2*

Let $h: (R, d) \longrightarrow (R, d)$ where $h(a) = \frac{1}{2}a$, and $d(a, b) = |a - b|$ for all $a, b \in R$. Then, the Mapping $h$ is generalized non expansive mapping. Since for each $a, b \in [0,1]$ then the following inequality holds

$$|h(x) - h(y)| \leq \left(1 - \frac{1}{2}\right)|x - y| + \frac{1}{2}\big|d\big(x, h\big(h(x)\big) - d(y, h(y)\big)\big|$$

*Example 2*

Let $h: (R, d) \longrightarrow (R, d)$ such that $h(x) = x^2$, where $d(x, y) = |x - y|$ for all $x, y \in R$.

If $a = \frac{2}{3}$ , $b = \frac{1}{3}$ , $x = 2$ and $y = 5$. Then it is clear that the mapping $h$ is not generalized, non-expansive mapping.

*Remark 2*

i. Every contraction mapping is generalized non expansive Mapping but the conversely is not true.

ii. If $a \in (0,1)$ and $b = 0$ in definition (2.1) then $h$ is contraction mapping.

iii. If $a, b = 0$ in definition (2.1) then $h$ is non expansive mapping.

*Theorem (Extended Fixed Point Theorem)*

Let $(M, d)$ be a complete metric space and $R: (M, d) \longrightarrow (M, d)$ continuous and generalized non expansive mapping with constants $a \in (0,1]$ and $b \in [0,1]$. Then the mapping $R$ has a unique fixed point $c \in M$. On the other hand,for any $a \in M$ we have the following:

$$\lim_{n\to\infty} R^n\,(a) = c \text{ and } d(R^n(a), c) \le \frac{(1-\breve{a})^n}{\breve{a}} d\big(x, R(x)\big).$$

Proof:

The proof itself gives a method for approximating the fixed point.

Let us proceed in the following steps:

*Step 1*

Let $a \in M$ Define a sequence $\langle a_n \rangle$ by $a_n = R(a_{n-1})$.For each $n \in N$,

$$d\big(R^n(a), R^{n+1}(a)\big) \le (1-\breve{a})d\big(R^{n-1}(a), R^n(a)\big) + b\big|d(a, R\big(R^n(a)\big) - d(x, R^{n+1}(x))\big|$$

$$= (1-\breve{a})d\big(R^{n-1}(a), R^n(a)\big)$$

$$\le (1-\breve{a})\ \big[(1-a)d\big(R^{n-2}(a), R^{n-1}(a)\big) + b\big|d(a, R\big(R^{n-2}(a)\big) - d(a, R^{n-1}(a))\big|\big]$$

Therefore,

$$d\big(R^n(a), R^{n+1}(a)\big) \le (1-\breve{a})^2 d\big(R^{n-2}(a), R^{n-1}(a)\big)$$

$$d\big(R^n(a), R^{n+1}(a)\big) \le (1-\breve{a})^n d\big(a, R(a)\big)$$

*Step 2*

For any $m > n$,we have

$$d\big(R^n(a), R^m(a)\big) \le d\big(R^n(a), R^{n+1}(a)\big) + d\big(R^{n+1}(a), R^{n+2}(a)\big) + \;\ldots.\ldots\ldots. + d\big(R^{m-1}(a), R^m(a)\big)$$

By step 1 we get,

$$d\big(R^n(a), R^m(a)\big) \le (1-\breve{a})^n d\big(a, R(a)\big) + (1-\breve{a})^{n+1} d\big(a, R(a)\big) + \;\ldots\ldots\ldots. + (1-\breve{a})^{m-1} d\big(a, R(a)\big)$$

$$\le (1-\breve{a})^n[1 + (1-\breve{a}) + \;\ldots\ldots\ldots.\; + (1-\breve{a})^{m-n-1}]d\big(a, R(a)\big)$$

$$\le (1-\breve{a})^n \sum_{k=0}^{m-n-1} (1-\breve{a})^k\, d\big(a, R(a)\big)$$

$$\le (1-\breve{a})^n \sum_{k=0}^{\infty} (1-\breve{a})^k\, d\big(a, R(a)\big)$$

$$= (1-\breve{a})^n \frac{1}{1-(1-\breve{a})} d\big(a, R(a)\big) = \frac{(1-\breve{a})^n}{\breve{a}} d\big(a, R(a)\big)$$

*Step 3*

For each $\epsilon > 0$, since $(1-\breve{a}) \in [0,1)$ we can find a large $k \in N$ such that

$$(1-\breve{a})^n < \frac{\epsilon\breve{a}}{d\big(a, R(a)\big)}$$

By step (2) we have,

$$d\big(R^n(a), R^m(a)\big) \le \frac{(1-\breve{a})^n}{\breve{a}} d\big(a, R(a)\big)$$

$$< \frac{\epsilon\breve{a}}{\breve{a}d\big(a, R(a)\big)} d\big(a, R(a)\big) \quad = \epsilon$$

Hence, $\langle R^n(a_n) \rangle$ is Cauchy sequence. But $M$ be a complete metric space, therefore it is convergent to a point $c \in X$ such that

$$\lim_{n\to\infty} R^n\,(a) = c$$

*Step 4*

To prove that $c$ is fixed point.

Since is $R$ continuous mapping then we have,

$$c = \lim_{n\to\infty} R^{n+1}(a) = \lim_{n\to\infty} R(R^n(a)) = R \lim_{n\to\infty} R^n(a) = R(c)$$

Therefore, $c$ is fixed point of $R$.

*Step 5*

We show that the fixed point is unique.

Suppose that there exists $a, b \in X$ such that $R(a) = a$ and $R(b) = b$

$$d(a,b) = d\big(R(a), R(b)\big) \leq (1-\breve{a})d(a,b) + b\big|d(a, R\big(R(a)\big) - d(b, R(b))\big|$$
$$= (1-\breve{a})d(a,b) + b|d(a, R(a)| < d(a,b)$$

Which is a contradiction. And hence, $d(a,b) = 0$. So, $a = b$.

**New Iterative Methods**

In this section we have introduced some iteration methods.

*First Method (Modified Picard Iteration Method)*

*Definition 1*

Let $M$ be a nonempty subset of normed space , $b_0 \in M$ be arbitrary and $0 \leq \breve{a}, \breve{b} \leq 1$ . If the sequence $\langle b_n \rangle$ satisfy the following:

$$b_{n+1} = b_0 + \int_{x_0}^{x} R(t, b_n(t))dt \qquad \text{if } n = 0,1.$$

$$\left.\begin{aligned} b_{n+1} &= (1-\breve{a})R_{n-1} + \breve{a}R(a_{n-1}) \\ a_n &= \big(1-\breve{b}\big)b_n + \breve{b}R(b_n) \end{aligned}\right\} \quad n \geq 2$$

$R(b_{n+1}) = b_n$ , $R(a_{n+1}) = a_n$ and $R = \int_{x_0}^{x} F(t, b_n(t))dt$

Then this iteration is called modified Picard iteration method.

*Second Method (Modified Three Step Iteration Method)*

*Definition 2*

Let $M$ be a nonempty subset of normed space , $y_0 \in M$ be arbitrary and $0 \leq a, b, c \leq 1$ . If the sequence$\langle b_n \rangle$ satisfy the following:

$$b_{n+1} = b_0 + \int_{x_0}^{x} F(t, b_n(t))dt \qquad \text{if } n = 0,1.$$

$$\left.\begin{aligned} y_{n+1} &= \breve{a}a_n + (1-\breve{a})R(b_{n-1}) \\ a_n &= \breve{b}R(b_{n-1}) + \big(1-\breve{b}\big)c_{n-1} \\ c_n &= \check{c}b_n + (1-\check{c})R(b_n) \end{aligned}\right\} \quad n \geq 2$$

$R(b_{n+1}) = b_n$ and $R = \int_{x_0}^{x} R(t, b_n(t))dt$

Then this iteration is called modified three step iteration method.

**Third Method (Extended Ishikawa Iteration Method)**

*Definition 3*

Let $M$ be a nonempty subset of normed space , Let $b_0 \in M$ be arbitrary and $0 \leq \breve{a}, \breve{b}, \check{c} \leq 1$ . If the sequence$\langle b_n \rangle$ satisfy the following:

$$b_{n+1} = b_0 + \int_{x_0}^{x} R(t, b_n(t))dt \qquad \text{if } n = 0,1.$$

$b_{n+1} = \breve{a}\, a_{n-2} + (1-\breve{a})\, R(b_{n-1})$ if $n = 2,4, \dots \dots$

$b_{n+1} = (1-\breve{b})\, a_{n-1} + \breve{b}\, R(b_{n-2})$ if $n = 3,5, \dots \dots$

Where, $a_n = (1-\check{c})b_{n+1} + \check{c}\; R(b_{n-1})$

$R(b_{n-1}) = b_n$ and $R = \int_{x_0}^{x} R(t, b_n(t))dt$

Then this iteration is called extended Ishikawa iteration method.

**Implementation**

In this study, we have compared among the numerical solutions of proposed method which is named modified Picard iteration method and the classical methods: Euler, modified Euler and the successive over- relaxation (SOR) methods during the following IVP:

*Example 1*

$$w' = -w; \qquad 0 \le t \le 1,$$

With initial condition $w(0) = 1$,

and the exact solution is $w(x) = e^{-x}$.

In Fig. 1, we have compared among the numerical solutions of the classical methods, Euler, modified Euler and the successive over-relaxation (SOR) methods together to the exact solution of the IVP in Example 1

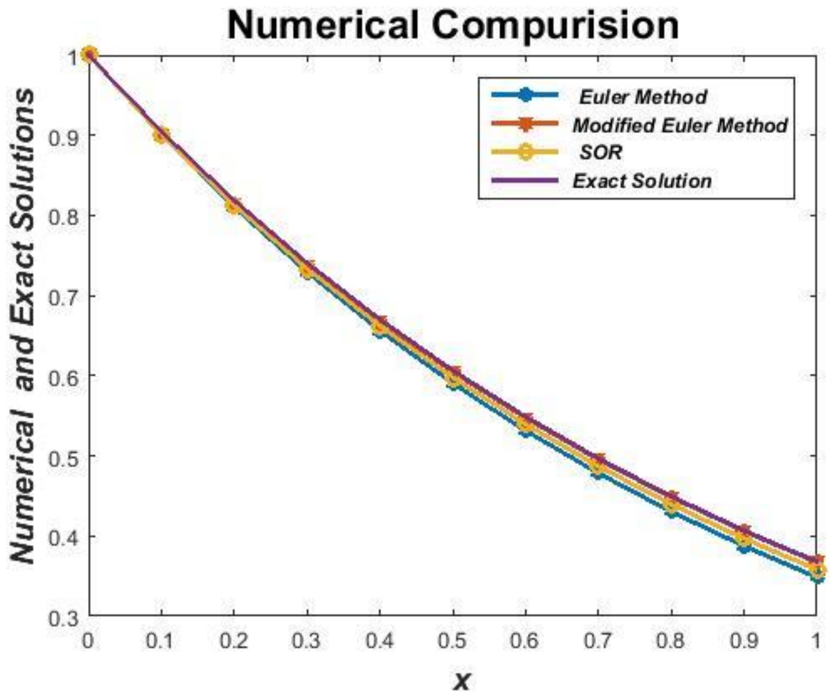


Fig. 1. Numerical Compurision

However, in Fig. 2, the absolute errors of the successive over-relaxation (SOR), Euler and modified Euler methods have been compared for the Example 1.

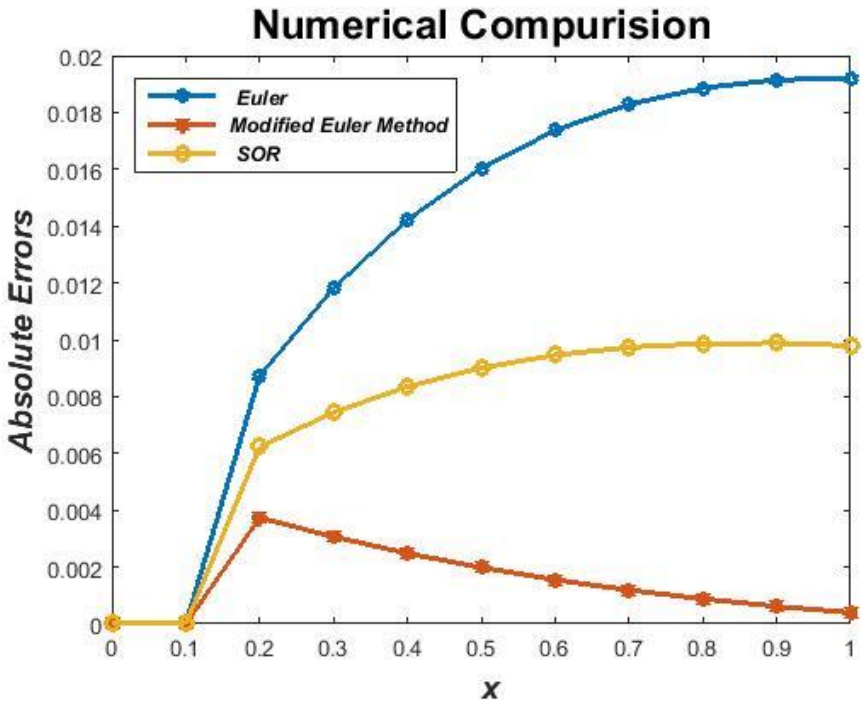


Fig. 2. Comparison of the absolute errors of the successive over-relaxation (SOR), Euler and modified Euler methods

While in Fig. 3, we have compared between the numerical solutions of Example 1 between proposed method which named modified Picard iteration method and Euler method of the IVP in Example 1.

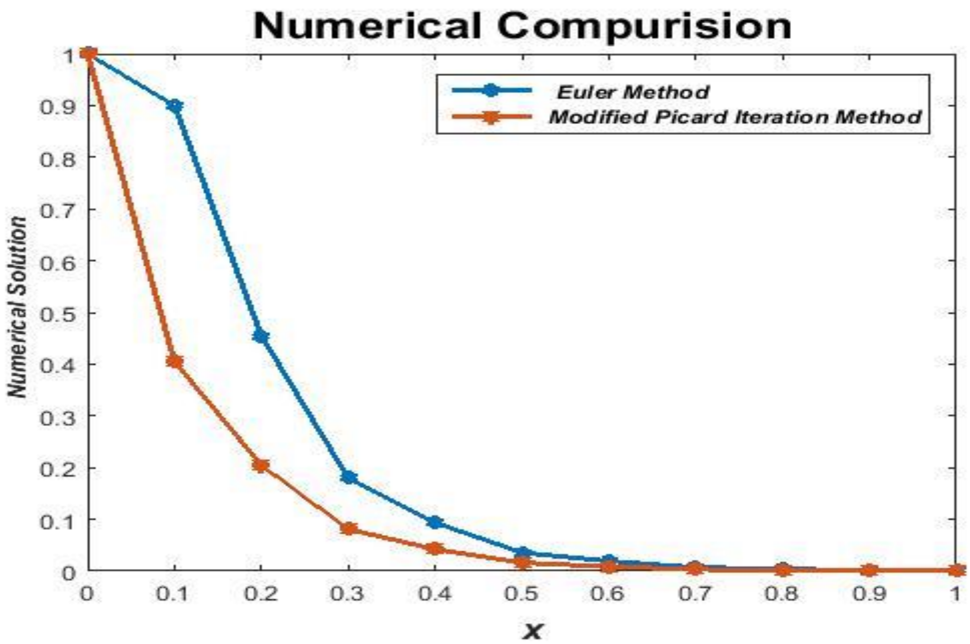


Fig. 3. Comparison of the numerical solutions of Example 1 between proposed method named modified Picard iteration method and Euler method of the IVP

**Conclusion**

In this paper, the definition of generalized non expansive mapping has been introduced. The new iterative methods for solving ODEs with initial condition have been analysed. We have compared the approximated results of derived methods with classical method. Also, the convergence analysis for approximation of fixed point of generalizes non expansive type mapping in suitable space has been discussed. Numerical comparison study using MATLAB software shows that the time complexity, accuracy and efficiency of the proposed method. However, the approximated solution is agreeing well with analytical solution for the tested problem.

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