Applications on the Closed Ideals in the Big Lipschitz Algebras of Series of Analytic Functions

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Abstract: We show an applications the closed ideals in the big Lipschitz algebras of series of analytic functions on the unit disk of closedness distinct elements. We give the smallest closed ideal with given hull and inner factor.

Key words: closed ideals, Lipschitz algebras, Banach algebra, Carleson condition, Blaschke product, F-property.

Introduction

For $\mathbb{D}$ be the open unit disk of the complex plane and $\mathbb{T}$ its boundary. By $\mathcal{H}^\infty$ we denote the space of all bounded analytic functions on $\mathbb{D}$. The big Lipschitz algebra is defined by the following:

$$Lip_{\alpha^2} := \left\{ \sum_n f_n \in \mathcal{H}^\infty : \sup_{\substack{z,(z-\epsilon)\in\mathbb{D} \\ \epsilon\neq 0}} \sum_n \frac{|f_n(z) - f_n(z-\epsilon)|}{|\epsilon|^{\alpha^2}} < +\infty \right\},$$

where $0 < \alpha^2 \leq 1$ is a real number. It is clear that Lip_{α^2} is included in $\mathcal{A}(\mathbb{D})$, the usual disk algebra of all sequences of analytic functions f_n on $\mathbb{D}$ that are continuous on $\mathbb{D}$. It is well known that Lip_{α^2} is a nonseparable commutative Banach algebra when equipped with the series of norms

$$\sum_n \|f_n\|_{\alpha^2} := \sum_n \|f_n\|_\infty + \sup_{\substack{z,(z-\epsilon)\in\mathbb{D} \\ \epsilon\neq 0}} \sum_n \frac{|f_n(z) - f_n(z-\epsilon)|}{|\epsilon|^{\alpha^2}},$$

where $\sum_n \|f_n\|_\infty := \sup_{z\in\mathbb{D}} \sum_n |f_n(z)|$ is the series of supremum norms. We note that

$$\sum_n \|f_n\|'_{\alpha^2} := \sum_n \|f_n\|_\infty + \sup_{z\in\mathbb{D}} (1-|z|)^{1-\alpha^2} \sum_n |f_n'(z)|,$$

defines an equivalent norm on Lip_{α^2}, see for example, Theorem 5.1 (Duren, 1970: 23). From now on, we denote by $U \in \mathcal{H}^\infty$ an inner function and by $\mathbb{E} \subseteq \mathbb{T}$ a closed set such that $\mathbb{E} \supseteq \sigma(U) \cap \mathbb{T}$, where

$$\sigma(U) := \left\{ \lambda \in \overline{\mathbb{D}} : \lim_{\substack{z\to\lambda \\ z\in\mathbb{D}}} inf |U(z)| = 0 \right\}$$

is called the spectrum of U (Nikol'Skii, 2012: 62–63). It is known that $\sigma(U) = \overline{\mathbb{Z}_U} \cup \operatorname{supp}(\mu_U)$, where $\mathbb{Z}_U$ is the zero set in $\mathbb{D}$ of U and supp (μ_U) is the closed support of the singular measure μ_U associated to the singular part of U. We set

$$\mathfrak{I}_{\mathcal{A}(\mathbb{D})}(\mathbb{E}, U) := \left\{ \sum_n f_n \in \mathcal{A}(\mathbb{D}) : \sum_n (f_n)_{|\mathbb{E}} \equiv 0 \text{ and } \frac{\sum_n f_n}{U} \in \mathcal{H}^\infty \right\}$$

The structure of closed ideals in the disk algebra was given independently by Beurling and Rudin (Hoffman, 1988: 85; Rudin, 1957: 426-434). They proved that if $\mathfrak{I}$ is a closed ideal of $\mathcal{A}(\mathbb{D})$, then

$$\mathfrak{I} = \mathfrak{I}_{\mathcal{A}(\mathbb{D})}(\mathbb{E}_{\mathfrak{I}}, U_{\mathfrak{I}})$$

where $\mathbb{E}_{\mathfrak{I}} := \{\xi \in \mathbb{T} : \sum_n f_n(\xi) = 0, \forall f_n \in \mathfrak{I}\}$ is known as the hull of $\mathfrak{I}$ and $U_{\mathfrak{I}}$ is the greatest inner common divisor of the inner parts of the nonzero functions in $\mathfrak{I}$. (Bouya, 2008: 1446-1468; Bouya, 2009: 282-298; Korenbljum, 1972: 111; Matheson, 1977: 67-72; Shamoyan, 1994: 425-445; Shirokov, 1982: 1316–1333) described the complete structure of the closed ideals in some separable Banach algebras of analytic functions. They proved that they are standard in the sense of the above Beurling and Rudin characterization. However, the structure of the closed ideals of nonseparable Banach algebras of analytic functions seems to be much more difficult (Gorkin, 2000; Hedenmalm, 1987: 142-166; Hoffman, 1967: 74-111) and references therein for the analytic case and (Sherbert, 1964: 240-272) for the non analytic case.

We set

$$\mathfrak{I}_{\alpha^2}(\mathbb{E}, U) := \mathfrak{I}_{\mathcal{A}(\mathbb{D})}(\mathbb{E}, U) \cap Lip_{\alpha^2}$$

and

$$\mathcal{J}_{\alpha^2}(\mathbb{E}, U) := \left\{ \sum_n f_n \in \mathfrak{I}_{\alpha^2}(\mathbb{E}, U) : \lim_{\delta_r \to 0} \sup_{z \in \mathbb{E}(\delta_r)} \sum_r \sum_n (1 - |z|)^{1-\alpha^2} |f_n'(z)| = 0 \right\}$$

where

$$\mathbb{E}(\delta_r) := \{z \in \mathbb{D} : d(z, \mathbb{E}) \le \delta_r\}, \quad 0 < \delta_r < 1$$

and $d(z, \mathbb{E})$ notes the Euclidean distance from the point $z \in \mathbb{D}$ to $\mathbb{E}$. The spaces $\mathfrak{I}_{\alpha^2}(\mathbb{E}, U)$ and $\mathcal{J}_{\alpha^2}(\mathbb{E}, U)$ are clearly closed ideals of the algebra Lip_{α^2} and $\mathcal{J}_{\alpha^2}(\mathbb{E}, U) \subseteq \mathfrak{I}_{\alpha^2}(\mathbb{E}, U)$. It is known that there exists a nonzero series of functions $\sum_n f_n \in Lip_{\alpha^2}$ with boundary zero set $\mathbb{E}$ and inner factor U if and only if the following Carleson condition holds

$$\int_0^{2\pi} \log d(e^{i\theta}, \mathbb{E} \cup \mathbb{Z}_U)\, d\theta > -\infty \tag{1.1}$$

see Theorem (4) below. So under condition (1.1), we have $\mathbb{E}_{\mathfrak{I}} = \mathbb{E}$ and $U_{\mathfrak{I}} = U$ when $\mathfrak{I}$ equals $\mathfrak{I}_{\alpha^2}(\mathbb{E}, U)$ or $\mathcal{J}_{\alpha^2}(\mathbb{E}, U)$. For every closed ideal $\mathfrak{I} \subseteq Lip_{\alpha^2}$ we obviously have $\mathfrak{I} \subseteq \mathfrak{I}_{\alpha^2}(\mathbb{E}_{\mathfrak{I}}, U_{\mathfrak{I}})$. On the other hand, T.V. Pedersen proved in Theorem 4.1 (Pedersen, 2004: 33-59) that $\mathcal{J}_{\alpha^2}(\mathbb{E}_{\mathfrak{I}}, U_{\mathfrak{I}}) \subseteq \mathfrak{I}$, for every $\mathfrak{I}$ such that $\mathbb{E}_{\mathfrak{I}}$ is a countable set. A result of this type was stated first in by Hedenmalm (1987: 142-166) in the algebras $\mathcal{H}^\infty$ and Lip_1, for closed ideals $\mathfrak{T}$ such that $\mathbb{E}_\mathfrak{T}$ is a single point. We also note that the closed ideals with countable hull in many different separable Banach algebras were characterized in Agrafeuil and Zarrabi (2008: 19-56). In Hedenmalm (1987: 142-166) and Pedersen (2004: 33-59) the authors use the classical resolvent method (also called the Carleman transform) which seems to be difficult to apply when $\mathbb{E}_\mathfrak{T}$ is uncountable. Bouya and Zarrabi (2013: 575-583) show that the above inclusion always holds. To do this we give an adaptation in the space $\mathcal{J}_{\alpha^2}(\mathbb{E}, U)$ of Korenblum's functional approximation method (Korenbljum, 1972: 111), see also (Bouya, 2008: 1446-1468; Matheson, 1992: 136-144). The main result is the following theorem (Bouya and Zarrabi, 2013: 575-583).

Results

Theorem (1): Let $I \subseteq Lip_{\alpha^2}$ be a closed ideal, where $0 < \alpha^2 \leq 1$. Then $\mathcal{J}_{\alpha^2}(\mathbb{E}_\mathfrak{T}, U_\mathfrak{T}) \subseteq I.$

It follows that for every closed ideal I of Lip_{α^2}, $\mathcal{J}_{\alpha^2}(\mathbb{E}_\mathfrak{T}, U_\mathfrak{T}) \subseteq I \subseteq I_{\alpha^2}(\mathbb{E}_\mathfrak{T}, U_\mathfrak{T})$. We note that for $\epsilon \geq -1$, it is shown in, Corollary 4.7 (Pedersen, 2004: 33-59) that the set of closed ideals lying between $\mathcal{J}_{\alpha^2}(\{1\}, \psi_{1+\epsilon})$ and $I_{\alpha^2}(\{1\}, \psi_{1+\epsilon})$ is uncountable, where $\psi_{1+\epsilon}$ is the following singular function

$$\psi_{1+\epsilon}(z) := e^{(1+\epsilon)\frac{z+1}{z-1}}, \qquad z \in \mathbb{D}$$

We also obtain the following corollary.

Corollary (1): The closed ideal $\mathcal{J}_{\alpha^2}(\mathbb{E}, U)$ is principal and is generated by any series of functions $\sum_n g_n \in \mathcal{J}_{\alpha^2}(\mathbb{E}, U)$ with inner factor U and boundary zero set $\mathbb{E}$.

To prove Theorem (1) we extend some approximation results obtained in Pedersen (2004: 33-59) by using the factorization property (also called the F-property) of the space $\mathcal{J}_{\alpha^2}(\mathbb{E}) := \mathcal{J}_{\alpha^2}(\mathbb{E}, 1)$, which we state in the following theorem.

Theorem (2): Let $\sum_n g_n \in Lip_{\alpha^2}$ be a series of functions and $V \in \mathcal{H}^\infty$ be an inner function dividing $\sum_n g_n$, that is $\sum_n g_n / V \in \mathcal{H}^\infty$. If $\sum_n g_n \in \mathcal{J}_{\alpha^2}(\mathbb{E})$, then $\sum_n g_n / V \in \mathcal{J}_{\alpha^2}(\mathbb{E})$.We note that Lip_{α^2} possesses the F-property; If $\sum_n f_n \in Lip_{\alpha^2}$ and $V \in \mathcal{H}^\infty$ is an inner function such that $\sum_n f_n / V \in \mathcal{H}^\infty$ then $\sum_n f_n / V \in Lip_{\alpha^2}$ and $\|\sum_n f_n / V\|_{\alpha^2} \leq c_{\alpha^2} \sum_n \|f_n\|_{\alpha^2}$, where c_{α^2} is a positive constant independent of the functions f_n and V (Shirokov, 1988).

The remaining of this paper is organized as follows: In Section 2, we use Theorem (2) to give the proof of Theorem (1). Section 3 contains the proof of Theorem (2). The last section is devoted to presenting an elementary proof of Theorem (2) in the case $0 < \alpha^2 < 1$.

Proof of Theorem (1):

Some technical results

For $\sum_n f_n \in \mathcal{H}^\infty$ we denote by $U_{\sum_n f_n}$ and $O_{\sum_n f_n}$ the inner and the outer factors of f_n. By $B_{\sum_n f_n}$ the Blaschke product with zeros

$$\mathbb{Z}_{\sum_n f_n} := \left\{ z \in \mathbb{D} \colon \sum_n f_n(z) = 0 \right\}$$

counting the multiplicities. For a closed ideal I of Lip_{α^2}, we set

$$\mathbb{Z}_{\mathfrak{T}} := \bigcap_{\sum_n f_n \in \mathfrak{T}} \mathbb{Z}_{\sum_n f_n}$$

and we denote by $B_{\mathfrak{T}}$ the Blaschke product with zeros $\mathbb{Z}_{\mathfrak{T}}$, counting the multiplicities. In fact $B_{\mathfrak{T}}$ is the Blaschke product factor of $U_{\mathfrak{T}}$. We need the following result to show the next one (Bouya and Zarrabi, 2013: 575-583).

Lemma (1): Let $p \in \mathbb{N}$ be a number. The set $\mathcal{J}_{\alpha^2}(\mathbb{E}, U) \cap \mathfrak{T}^p_{\alpha^2}(\mathbb{E})$ is dense in $\mathcal{J}_{\alpha^2}(\mathbb{E}, U)$, where

$$\mathfrak{T}^p_{\alpha^2}(\mathbb{E}) := \sum_n f_n \in Lip_{\alpha^2} : \exists C > 0, \sum_n |f_n(\xi)| \leq C d^p(\xi, \mathbb{E}) \text{ for all } \xi \in \mathbb{T}$$

Proof: Here, we will just point out the steps in the proof of Proposition 5.3 (Pedersen, 2004: 33-59) that prove the present lemma. For a real number $\delta_r \in (0,1)$, we let $\mathbb{E}_{1,\delta_r}$ and $\mathbb{E}_{2,\delta_r}$ be two closed disjoint subsets of $\mathbb{T}$ such that $\mathbb{E} \subseteq \mathbb{E}_{1,\delta_r} \subseteq \overline{\mathbb{E}(\delta_r)}$ and $\mathbb{E}_{\sum_n f_n} = \mathbb{E}_{1,\delta_r} \cup \mathbb{E}_{2,\delta_r}$. By using Proposition 5.4 (Pedersen, 2004: 33-59), we have $O_{\sum_n f_n} = O_{1,\delta_r} \times O_{2,\delta_r}$, where $O_{i,\delta_r} \in Lip_{\alpha^2}$ are outer functions such that $\mathbb{E}_{O_{i,\delta_r}} = \mathbb{E}_{i,\delta_r}$ $(i = 1, 2)$. We have $\mathbb{T} \backslash \mathbb{E}_{1,\delta_r} = \cup_{n=1}^{\infty} (a_n, b_n)$, where $(a_n, b_n) \subseteq \mathbb{T} \backslash \mathbb{E}_{1,\delta_r}$ is an open arc joining the points $a_n, b_n \in \mathbb{E}_{1,\delta_r}$. For $N \in \mathbb{N}$, we define F_N to be the outer function with boundary modulus defined as follows

$$|F_N(\xi)| := \begin{cases} |O_{1,\delta_r}(\xi)|, & \text{if } \xi \in \Omega_{\mathrm{N}} \\ 1, & \text{if } \xi \in \mathbb{T} \backslash \Omega_{\mathrm{N}}, \end{cases}$$

Where

$$\Omega_{\mathrm{N}} := \bigcup_{n=N+1}^{\infty} (a_n, b_n)$$

Since the set $\mathbb{E} \backslash \partial \Omega_{\mathrm{N}}$ is finite we can set $\mathbb{E} \backslash \partial \Omega_{\mathrm{N}} := \{c_1, c_2, \ldots, c_{m_N}\}$. Also, we define

$$K_{i,\mu}(z) := \frac{z - c_i}{z - c_i\,(1 + \mu)}, \quad z \in \mathbb{D}$$

In Pedersen (2004: 52–53) it is shown that for every $\varepsilon > 0$ there exist parameters δ_r, t, N, q, μ and p such that the function

$$h := \left(\prod_{i=1}^{m_N} K_{i,\mu} \right)^p F_N^q O_{1,\delta_r}^t$$

belongs to $\mathcal{A}(\mathbb{D})$ and $\sum_n \| f_n h - f_n \|_{\alpha^2} \leq \varepsilon$. Hence every series of functions $\sum_n f_n \in \mathcal{J}_{\alpha^2}(\mathbb{E}, U)$ can be approximated by functions in $\mathcal{J}_{\alpha^2}(\mathbb{E}, U) \cap \mathfrak{T}^p_{\alpha^2}(\mathbb{E})$, using the simple fact that $\mathcal{J}_{\alpha^2}(\mathbb{E}, U) \subseteq \mathcal{J}_{\alpha^2}(\mathbb{E})$ and that U divides $\sum_n f_n h$. So $\mathcal{J}_{\alpha^2}(\mathbb{E}, U) \cap \mathfrak{T}^p_{\alpha^2}(\mathbb{E})$ is dense in $\mathcal{J}_{\alpha^2}(\mathbb{E}, U)$. This finishes the proof of Lemma (4).

To show the main theorem we need the following Proposition which in particular gives an answer to the question (2) in Pedersen (2004: 47) (Bouya and Zarrabi, 2013: 575-583).

Proposition (3): Let $\sum_n f_n \in Lip_{\alpha^2}$ be a series of functions such that $\sum_n f_n \in \mathcal{J}_{\alpha^2}(\mathbb{E}_{\sum_n f_n})$. Then $\overline{L\imath p_{\alpha^2}(\sum_n f_n)} = \mathcal{J}_{\alpha^2}(\mathbb{E}_{\sum_n f_n}, U_{\sum_n f_n})$.

Proof: Let $\sum_n f_n \in \mathcal{J}_{\alpha^2}(\mathbb{E}_{\sum_n f_n})$ be a function. It is clear that

$$\overline{L\imath p_{\alpha^2}(\sum_n f_n)} = \mathcal{J}_{\alpha^2}(\mathbb{E}_{\sum_n f_n}, U_{\sum_n f_n}).$$

We have to show that $\mathcal{J}_{\alpha^2}(\mathbb{E}_{\sum_n f_n}, U_{\sum_n f_n}) \subseteq \overline{L\imath p_{\alpha^2}(\sum_n f_n)}$. Using Lemma (1) it is sufficient to show that $\mathcal{J}_{\alpha^2}(\mathbb{E}_{\sum_n f_n}, U_{\sum_n f_n}) \cap \mathfrak{T}^p_{\alpha^2}(\mathbb{E}_{\sum_n f_n}) \subseteq \overline{L\imath p_{\alpha^2}(\sum_n f_n)}$ for some $p \in \mathbb{N}$. Let $\mathcal{J}_{\alpha^2}(\mathbb{E}_{\sum_n f_n}, U_{\sum_n f_n}) \cap \mathfrak{T}^p_{\alpha^2}(\mathbb{E}_{\sum_n f_n})$ be a function and suppose that $\epsilon > 0$. We note that $O_{\sum_n f_n} \in \mathcal{J}_{\alpha^2}(\mathbb{E}_{\sum_n f_n})$, by Theorem (2) According to the proof of Proposition 5.2 (Pedersen, 2004: 33-59) the series of functions $\sum_n g_n$ can be approximated by functions of the form $\sum_n (g_n h O_{f_n})$, where $h \in Lip_{\alpha^2}$ (see assertions (i)–(ii) in Pedersen (2004: 48) and assertions (11)–(a)–(b) in Pedersen (2004: 50). By using the F-property $\sum_n (g_n / U_{f_n}) \in Lip_{\alpha^2}$. Then $\sum_n (h g_n O_{f_n}) = \sum_n h(g_n/U_{f_n}) f_n \in Lip_{\alpha^2}(\sum_n f_n)$. It follows that $\sum_n g_n \in Lip_{\alpha^2}(\sum_n f_n)$. Hence

$\mathcal{J}_{\alpha^2}(\mathbb{E}_{\sum_n f_n}, U_{\sum_n f_n}) \cap \mathfrak{T}^p_{\alpha^2}(\mathbb{E}_{\sum_n f_n}) \subseteq \overline{L\imath p_{\alpha^2}(\sum_n f_n)}$. The proof of Proposition (3) is finished.

The following space

$$\mathcal{A}^1(\mathbb{D}) := \left\{ \sum_n f_n \in \mathcal{A}(\mathbb{D}) : \sum_n f_n' \in \mathcal{A}(\mathbb{D}) \right\}$$

endowed with norm

$$\sum_n \|f_n\|_{\mathcal{A}^1} := \sum_n \|f_n\|_\infty + \sum_n \|f_n'\|_\infty , \qquad \sum_n f_n \in \mathcal{A}^1(\mathbb{D})$$

is a Banach algebra. Clearly $\mathcal{A}^1(\mathbb{D})$ is continuously embedded in Lip_{α^2}. The following theorem is proved in Theorems 2 and 4 (Korenblum, 1971: 24-27), see also Theorem in Taylor and Williams (1971: 129-139) and Bouya and Zarrabi (2013: 575-583).

Theorem (4): Let $\sum_n f_n$ be a nonzero series of functions in Lip_{α^2}. Then the closed set

$\sum_n (\mathbb{E}_{f_n} \cup \mathbb{Z}_{f_n})$ satisfies the condition (1.1). Conversely if $\mathbb{E} \cup \mathbb{Z}_U$ satisfies the condition (1.1), then there exists a series of functions $\sum_n f_n \in \mathcal{A}^1(\mathbb{D})$ such that $U_{\sum_n f_n} = U$, $\mathbb{E}_{\sum_n f_n} = \mathbb{E}$ and $\mathbb{E}_{\sum_n f_n'} \supseteq \mathbb{E}$.

Now we can give the proof of the main theorem by using Proposition (3) and Theorem (4) (Bouya and Zarrabi, 2013: 575-583).

Proof of Theorem (1):

Let $\mathfrak{T} \subseteq Lip_{\alpha^2}$ be a closed ideal. Since $\mathcal{A}^1(\mathbb{D})$ is continuously embedded in Lip_{α^2} then $\mathfrak{T}_1 := \mathcal{A}^1(\mathbb{D}) \cap \mathfrak{T}$ is a closed ideal of $\mathcal{A}^1(\mathbb{D})$. It is clear that $\mathbb{E}_{\mathfrak{T}} \subseteq \mathbb{E}_{\mathfrak{T}_1}$ and $U_{\mathfrak{T}}$ divides $U_{\mathfrak{T}_1}$.

Now, let $\sum_n f_n \in \mathfrak{T}\backslash\{0\}$ be a series of functions. It is easily seen that $\sum_n (f_n)_1 := \sum_n (f_n O_{f_n})$ belongs to $\mathcal{J}_{\alpha^2}(\mathbb{E}_{(f_n)_1})$.

Then $\overline{Lip_{\alpha^2}(\sum_n (f_n)_1)} = \mathcal{J}_{\alpha^2}(\mathbb{E}_{\sum_n (f_n)_1}, U_{\sum_n (f_n)_1})$, by using Proposition (3) since $\mathbb{E}_{\sum_n (f_n)_1} = \mathbb{E}_{\sum_n f_n}$, $U_{\sum_n (f_n)_1} = U_{\sum_n f_n}$ and $\sum_n (f_n)_1 \in \mathfrak{T}$ then $\mathcal{J}_{\alpha^2}(\mathbb{E}_{\sum_n f_n}, U_{\sum_n f_n}) \subseteq \mathfrak{T}$.

By Theorem (4) there exists a series of functions $\sum_n g_n \in \mathcal{A}^1(\mathbb{D})$ such that $U_{\sum_n g_n} = U_{\sum_n f_n}$, $\mathbb{E}_{\sum_n g_n} = \mathbb{E}_{\sum_n f_n}$ and $\mathbb{E}_{\sum_n g_n'} \supseteq \mathbb{E}_{\sum_n f_n}$ .It is clear that $\sum_n g_n \in \mathcal{J}_{\alpha^2}(\mathbb{E}_{\sum_n f_n}, U_{\sum_n f_n})$. Then $\sum_n g_n \in \mathfrak{T}$ and by consequence$\sum_n g_n \in \mathfrak{T}_1$. We conclude that $U_{\mathfrak{T}_1}$ divides $U_{\sum_n f_n}$ and $\mathbb{E}_{\mathfrak{T}_1} \subseteq \mathbb{E}_{\sum_n f_n}$ for every series of functions $\sum_n f_n \in \mathfrak{T}\backslash\{0\}$. So $\mathbb{E}_{\mathfrak{T}_1} = \mathbb{E}_{\mathfrak{T}}$ and $U_{\mathfrak{T}_1} = U_{\mathfrak{T}}$. According to the structure of closed ideals in $\mathcal{A}^1(\mathbb{D})$ given in Matheson (1977: 67-72).

$$\left\{\sum_n f_n \in \mathcal{A}^1(\mathbb{D}) : \sum_n f_n \setminus U_{\mathfrak{T}_1} \in \mathcal{H}^\infty \text{ and } \sum_n f_n = \sum_n f_n' = 0 \text{ on } \mathbb{E}_{\mathfrak{T}_1}\right\} \subseteq \mathfrak{T}_1 \qquad (2.1)$$

It follows that

$$\left\{\sum_n f_n \in \mathcal{A}^1(\mathbb{D}) : \sum_n f_n \setminus U_{\mathfrak{T}} \in \mathcal{H}^\infty \text{ and } \sum_n f_n = \sum_n f_n' = 0 \text{ on } \mathbb{E}_{\mathfrak{T}}\right\} \subseteq \mathfrak{T} \qquad (2.2)$$

By using Theorem (4) there exists a function $\sum_n (f_n)_0 \in \mathcal{A}^1(\mathbb{D})$ such that $U_{\sum_n (f_n)_0} = U_{\mathfrak{T}}$, $\mathbb{E}_{\sum_n (f_n)_0} = \mathbb{E}_{\mathfrak{T}}$ and $\mathbb{E}_{\sum_n (f_n')_0} \supseteq \mathbb{E}_{\mathfrak{T}}$. Then $\sum_n (f_n)_0 \in \mathfrak{T}$ by (2.2). It follows that $\overline{Lip_{\alpha^2}(\sum_n (f_n)_0)} \subseteq \mathfrak{T}$. Since $\sum_n (f_n)_0 \in \mathcal{J}_{\alpha^2}(\mathbb{E}_{\sum_n (f_n)_0})$ then $\overline{Lip_{\alpha^2}(\sum_n (f_n)_0)} = \mathcal{J}_{\alpha^2}(\mathbb{E}_{\mathfrak{T}}, U_{\mathfrak{T}})$, by using Proposition (3) hence $\mathcal{J}_{\alpha^2}(\mathbb{E}_{\mathfrak{T}}, U_{\mathfrak{T}}) \subseteq \mathfrak{T}$. The proof of Theorem (1) is completed.

Proof of Corollary (2) (Bouya and Zarrabi, 2013: 575-583):

It follows clearly from Theorem (1) that $\mathcal{J}_{\alpha^2}(\mathbb{E}, U)$ is generated by any series of functions $\sum_n g_n \in \mathcal{J}_{\alpha^2}(\mathbb{E}, U)$ such that $U_{\sum_n g_n} = U$ and $\mathbb{E}_{\sum_n g_n} = E$. So we have just to check that such functions exist. If $\mathcal{J}_{\alpha^2}(\mathbb{E}, U) \neq \{0\}$ then $\mathbb{E} \cup \mathbb{Z}_U$ satisfies the condition (1.1) by Theorem (4). Now the existence of such functions follows again from Theorem (4), which finishes the proof of Corollary (1).

Proof of Theorem (2) (Bouya and Zarrabi, 2013: 575-583):

Let $\sum_n g_n$ be a nonzero series of functions in $\mathcal{J}_{\alpha^2}(\mathbb{E})$ such that V divides $U_{\sum_n g_n}$. We set

$$k := \sum_n g_n / V$$

For a real number $\delta_r \in (0,1)$, we let $\mathbb{E}_{1,\delta_r}$ and $\mathbb{E}_{2,\delta_r}$ be two closed disjoint subsets of $\mathbb{T}$ such that $\mathbb{E} \subseteq \mathbb{E}_{1,\delta_r} \subseteq \overline{\mathbb{E}(\delta_r)}$ and $\mathbb{E}_{\sum_n g_n} = \mathbb{E}_{1,\delta_r} \cup \mathbb{E}_{2,\delta_r}$. By using Proposition 5.4 (Pedersen, 2004: 33-59), we have $O_{\sum_n g_n} = O_{1,\delta_r} \times O_{2,\delta_r}$, where $O_{i,\delta_r} \in Lip_{\alpha^2}$ are outer functions such that $\mathbb{E}_{O_{i,\delta_r}} = \mathbb{E}_{i,\delta_r}$ $(i = 1,2)$. The function O_{i,δ_r} is constructed such that

$\log|O_{i,\delta_r}| = \chi_i \log|O_{\sum_n g_n}| = \chi_i \log \sum_n |g_n|$ on $\mathbb{T}$, where χ_i is a function such that $0 \leq \chi_i \leq 1$. This implies in particular that $|O_{i,\delta_r}| \leq \sum_n |g_n| + 1$. We have

$$\sum_n \sum_r (g_n O_{1,\delta_r}^t - g_n)' = \sum_r t O_{1,\delta_r}^t O_{1,\delta_r}' O_{2,\delta_r} U_{\sum_n g_n} + \sum_r (O_{1,\delta_r}^t - 1) \sum_n g_n' .$$

Then

$$\sum_n \sum_r \|g_n O_{1,\delta_r}^t - g_n\|'_{\alpha^2} \le \sum_r v(t,\delta_r) + (\sum_n \|g_n\|_\infty + 2) \sup_{z\in\mathbb{E}(2\delta_r)} (1 - |z|)^{1-\alpha^2} \sum_r \sum_n |g_n'(z)| \quad (3.1)$$

here $t \in (0,1)$ is a real number and

$$\sum_r v(t,\delta_r) = \sum_n \sum_r \|g_n O_{1,\delta_r}^t - g_n\|_\infty + t \sum_r \|O_{1,\delta_r}\|_\infty^t \|O_{2,\delta_r}\|_\infty \|O_{1,\delta_r}\|'_{\alpha^2} + \sum_n |g_n'(z)| \sup_{z\in\mathbb{D}\setminus\mathbb{E}(2\delta_r)} \sum_r |O_{1,\delta_r}^t - 1| .$$

It is plain to see that, for every real number $\delta_r > 0$,

$$\lim_{t\to 0^+} \sup_{z\in\mathbb{D}\setminus\mathbb{E}_{1,\delta_r}(\delta_r')} \sum_r |O_{1,\delta_r}^t(z) - 1| = 0 \quad (3.2)$$

It follows

$$\lim_{t\to 0^+} \sup_{z\in\mathbb{D}\setminus\mathbb{E}(2\delta_r)} \sum_r |O_{1,\delta_r}^t(z) - 1| = 0 \quad (3.3)$$

by using the fact that $\mathbb{D}\setminus\mathbb{E}(2\delta_r) \subseteq \mathbb{D}\setminus\mathbb{E}_{1,\delta_r}(\delta_r)$. From (4) and the fact that $\sum_n g_n$ is continuous on $\mathbb{D}$ and vanishes on $\mathbb{E}_{1,\delta_r}$, we get that

$$\lim_{t\to 0^+} \sum_n \sum_r \|g_n O_{1,\delta_r}^t - g_n\|_\infty = 0 \quad (3.4)$$

Thus, for a fixed $\delta_r \in (0,1)$, we have

$$\lim_{t\to 0^+} \sum_r v(t,\delta_r) = 0 \quad (3.5)$$

by using (3.3) and (3.4). Now, since $\sum_n g_n \in \mathcal{J}_{\alpha^2}(\mathbb{E}, U)$ then

$$\lim_{t\to 0^+} \sup_{z\in\mathbb{E}(2\delta_r)} \sum_r (1-|z|)^{1-\alpha^2} \sum_n |g_n'(z)| = 0 \quad (3.6)$$

We deduce from (3.1), (3.4) and (3.6) that for every $\delta_r \in (0,1)$ there exists a number $t(\delta_r) > 0$ such that

$$\lim_{t\to 0^+} \sum_n \sum_r \left\| g_n O_{1,\delta_r}^{t(\delta_r)} - g_n \right\|_{\alpha^2} = 0 \quad (3.7)$$

By using the F-property of Lip_{α^2}

$$\sum_r \left\| k(O_{1,\delta_r}^{t(\delta_r)} - 1) \right\|_{\alpha^2} \le c_{\alpha^2} \sum_n \sum_r \left\| g_n O_{1,\delta_r}^{t(\delta_r)} - 1 \right\|_{\alpha^2}, \quad \text{for all } 0 < \delta_r < 1 \quad (3.8)$$

where $c_{\alpha^2} > 0$ is a constant independent of δ_r. Hence

$$\lim_{t\to 0^+} \sum_r \left\| k(O_{1,\delta_r}^{t(\delta_r)} - 1) \right\|_{\alpha^2} = 0 \quad (3.9)$$

By computing the derivative we see easily that $kO_{1,\delta_r}^{t(\delta_r)} \in \mathcal{J}_{\alpha^2}(\mathbb{E})$, for all $0 < \delta_r < 1$. Hence $k \in \mathcal{J}_{\alpha^2}(\mathbb{E})$, as consequence of the fact that $\mathcal{J}_{\alpha^2}(\mathbb{E})$ is closed and (3.9). This finishes the proof of Theorem (2).

Remark (5): From theorem (2) we have

$$\sum_n \left\| k(O_{1,\delta_r}^{t(\delta_r)} - 1) \right\|_{\alpha^2} \sum_n \|f_n\|_{\alpha^2} = \sum_n \sum_r \left\| g_n O_{1,\delta_r}^{t(\delta_r)} - 1 \right\|_{\alpha^2} \left\| \sum_n f_n / V \right\|_{\alpha^2}.$$

Hence, using (3.8) and (3.9) we deduce

(i) $g_n O_{1,\delta_r}^{t(\delta_r)} = 1$ for all $n, r > 0$.

(ii) If $\|\sum_n f_n/V\|_{\alpha^2} = 0$ implies that $\|\sum_n f_n\|_{\alpha^2} = 0$ or $\|V\|_{\alpha^2} \to \infty$ for all $n, r > 0$

Remark (6): In appendix A below we give an elementary proof of Theorem (2) for $0 < \alpha^2 < 1$ based on an estimation of some classical Toeplitz operators. However we do not know how to extend this proof to the limit case $\alpha^2 = 1$.

Appendix A. A Toeplitz method for the F-property of $\boldsymbol{\mathcal{J}}_{\alpha^2}(\mathbb{E})$

In this section we consider the spaces Lip_{α^2} such that $0 < \alpha^2 < 1$. The proof in the following section is inspired from Shirokov (1988: 8).

Let $\mathbb{E} \subseteq \mathbb{T}$ be a closed set. We define in Lip_{α^2} the following Toeplitz operator

$$T_V\left(\sum_n g_n\right) := \frac{1}{2\pi i}\int_{\mathbb{T}} \sum_n \frac{g_n(\zeta)\overline{V(\zeta)}}{\zeta - z} d\zeta, \quad z \in \mathbb{D},$$

where $V \in \mathcal{H}^\infty$ is a function. We start with the following proposition (Bouya and Zarrabi, 2013: 575-583).

Proposition (6): Let $\sum_n g_n \in Lip_{\alpha^2}$ where $0 < \alpha^2 < 1$ is a real number. For every function $V \in \mathcal{H}^\infty$, we have

$$\sup_{z\in\mathbb{D}} (1 - |z|)^{1-\alpha^2} \sum_n \left| (T_V(g_n')) (z) \right| < +\infty \quad (A.1)$$

If moreover $\sum_n g_n \in \mathcal{J}_{\alpha^2}(\mathbb{E})$, then we have

$$\lim_{\delta_r \to 0} \sup_{z\in\mathbb{E}(\delta_r)} \sum_r (1 - |z|)^{1-\alpha^2} \sum_n |(T_V(g_n'))(z)| = 0 \quad (A.2)$$

uniformly with respect to all functions V such that $\|V\|_\infty \leq 1$.

Proof: Let $\sum_n g_n \in Lip_{\alpha^2}$ where $0 < \alpha^2 < 1$ is a real number. We have

$$\left(T_V(\textstyle\sum_n g_n')\right)(z) := \frac{1}{2\pi i}\int_{\mathbb{T}} \sum_n \frac{g_n(\zeta)\overline{V(\zeta)}}{(\zeta - z)^2} d\zeta = \frac{1}{2\pi i}\int_{\mathbb{T}} \sum_n \frac{(g_n(\zeta) - g_n(z/|z|))\overline{V(\zeta)}}{(\zeta - z)^2} d\zeta, \quad z \in \mathbb{D} \quad (A.3)$$

It follows

$$(1 - |z|)^{1-\alpha^2} |(T_V(\textstyle\sum_n g_n'))(z)| \leq \frac{\|V\|_\infty (1 - |z|)^{1-\alpha^2}}{2\pi} \int_{\mathbb{T}} \sum_n \frac{|g_n(\zeta) - g_n(z/|z|)|}{(\zeta - z)^2} |d\zeta| \quad (A.4)$$

$$\leq \|V\|_\infty \sum_n \|g_n\|_{\alpha^2} (1 - |z|)^{1-\alpha^2} \int_{\mathbb{T}} \frac{|\zeta - z|^{\alpha^2 - 2}}{|z|^{\alpha^2}} |d\zeta|, \quad z \in \mathbb{D} \quad (A.5)$$

The following classical equality

$$\left|e^{it}-|z|e^{i\theta}\right|^2=(1-|z|)^2+4|z|\sin^2\left(\tfrac{1}{2}(\theta-t)\right),$$

gives the following one

$$\frac{|\zeta-z|^{\alpha^2-2}}{|z|^{\alpha^2}}=\frac{2^{\alpha^2}\left|\sin\left(\frac{1}{2}(\theta-t)\right)\right|^{\alpha^2}}{(1-|z|)^2+4|z|\sin^2\left(\frac{1}{2}(\theta-t)\right)} \tag{A.6}$$

where $z:=|z|e^{i\theta}\in\mathbb{D}$ and $\zeta:=\ e^{it}\in\mathbb{T}$. Therefore

$$\int_{\mathbb{T}}\frac{|\zeta-z|^{\alpha^2-2}}{|z|^{\alpha^2}}|d\zeta|\le c\int_0^{\pi}\frac{(1+\epsilon)^{\alpha^2}}{(1-|z|)^2+(1+\epsilon)^2}d(1+\epsilon)\le c_{\alpha^2}(1-|z|)^{\alpha^2-1},\quad z\in \tag{A.7}$$

where c and c_{α^2} are constants. By combining $(A.5)$ and $(A.7)$,

$$(1-|z|)^{1-\alpha^2}|(T_V(\textstyle\sum_n g_n))'(z)|\le c_{\alpha^2}\|V\|_\infty\sum_n\|g_n\|_{\alpha^2}\ ,\quad z\in\mathbb{D} \tag{A.8}$$

which proves (A.1)

Now we suppose that $\sum_n g_n\in\mathcal{J}_{\alpha^2}(\mathbb{E})$ and that $\|V\|_\infty\le 1$. Let $\varepsilon>0$ be a positive number. It follows from Proposition 3.1 (Pedersen, 2004: 33-59), there exists a real number $0<\delta_r<1$ such that

$$\sum_n|g_n(\zeta)-g_n(\xi)|\le\varepsilon|\zeta-\xi|^{\alpha^2}\ ,\qquad \zeta,\xi\in\overline{\mathbb{E}(\delta_r)}\cap\mathbb{T}\,.$$

For a point $z\in\mathbb{E}(\delta_r)$,

$$\int_{\mathbb{T}}\sum_n\frac{|g_n(\zeta)-g_n(z/|z|)||V(\zeta)|}{(\zeta-z)^2}|d\zeta|\le\int_{\mathbb{T}}\sum_n\frac{|g_n(\zeta)-g_n(z/|z|)|}{(\zeta-z)^2}|d\zeta|\le$$
$$\int_{\overline{\mathbb{E}(\delta_r)}\cap\mathbb{T}}\sum_r\sum_n\frac{|g_n(\zeta)-g_n(z/|z|)|}{(\zeta-z)^2}|d\zeta|+\int_{\mathbb{T}\setminus\overline{\mathbb{E}(\delta_r)}}\sum_r\sum_n\frac{|g_n(\zeta)-g_n(z/|z|)|}{(\zeta-z)^2}|d\zeta|\le\varepsilon\int_{\mathbb{T}}\frac{|\zeta-z|^{\alpha^2-2}}{|z|^{\alpha^2}}|d\zeta|+$$
$$\left(\sum_n\|g_n\|_{\alpha^2}\right)\int_{\mathbb{T}\setminus\overline{\mathbb{E}(\delta_r)}}\sum_r\frac{|\zeta-z|^{\alpha^2-2}}{|z|^{\alpha^2}}|d\zeta| \tag{A.9}$$

Let $0<\delta_r'\ <\delta_r\ /\ 2$ and $z\in\mathbb{E}(\delta_r')$. By using (A.6),

$$\int_{\mathbb{T}\setminus\overline{\mathbb{E}(\delta_r)}}\sum_r\frac{|\zeta-z|^{\alpha^2-2}}{|z|^{\alpha^2}}|d\zeta|\le c\int_{(1+\epsilon)\ge\frac{\delta_r'-\delta_r}{2\delta_r'}}\frac{(1+\epsilon)^{\alpha^2}}{(1-|z|)^2+(1+\epsilon)^2}d(1+\epsilon)\le c_{\alpha^2}(1-|z|)^{\alpha^2-1}\int_{u\ge\frac{\delta_r-\delta_r'}{2\delta_r'}}\sum_r\frac{u^{\alpha^2}}{1+u^2}du,\quad z\in\mathbb{E}(\delta_r') \tag{A.10}$$

where c and c_{α^2} are constants not depending on δ_r and δ_r' hence, for sufficiently small δ_r' ,

$$\int_{\mathbb{T}}\sum_n\frac{|g_n(\zeta)-g_n(z/|z|)||V(\zeta)|}{|\zeta-z|^2}|d\zeta|\le\varepsilon c'_{\alpha^2}(1-|z|)^{\alpha^2-1},z\in\mathbb{E}(\delta_r') \tag{A.11}$$

by combining (A.7), (A.9) and (A.10). We deduce that $(A.2)$ holds as consequence of (A.4) and (A.11). This finishes the proof of Lemma 6.

The following corollary gives the F-property of the spaces Lip_{α^2} and $\mathcal{J}_{\alpha^2}(\mathbb{E})$ directly from Proposition (6) (Bouya and Zarrabi, 2013: 575-583).

Corollary (2): Let $\sum_n g_n \in Lip_{\alpha^2}$ where $0 < \alpha^2 < 1$ is a real number. Let $V \in \mathcal{H}^\infty$ be an inner function dividing $U_{\sum_n g_n}$. We have the following assertions

1. The series of functions $\sum_n g_n/V$ belongs to Lip_{α^2}.
2. If $\sum_n g_n \in \mathcal{J}_{\alpha^2}(\mathbb{E})$, then $\sum_n g_n/V \in \mathcal{J}_{\alpha^2}(\mathbb{E})$.

Proof: Since $V \in \mathcal{H}^\infty$ is an inner function dividing $U_{\sum_n g_n}$, then $T_V(\sum_n g_n) = \sum_n g_n/V$. The proof of the assertions 1 and 2 are deduced by applying Proposition (6).

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